

Flash Cards

to accompany

A First Course in Linear Algebra

by

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Definition CNE Complex Number Equality

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The complex numbers $\alpha = a + bi$ and $\beta = c + di$ are **equal**, denoted $\alpha = \beta$, if $a = c$ and $b = d$.

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Definition CNA Complex Number Addition

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The **sum** of the complex numbers $\alpha = a + bi$ and $\beta = c + di$, denoted $\alpha + \beta$, is $(a + c) + (b + d)i$.

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The **product** of the complex numbers $\alpha = a + bi$ and $\beta = c + di$, denoted $\alpha\beta$, is $(ac - bd) + (ad + bc)i$.

Theorem PCNA Properties of Complex Number Arithmetic

The operations of addition and multiplication of complex numbers have the following properties.

- **ACCN Additive Commutativity, Complex Numbers** For any $\alpha, \beta \in \mathbb{C}$, $\alpha + \beta = \beta + \alpha$.
- **MCCN Multiplicative Commutativity, Complex Numbers** For any $\alpha, \beta \in \mathbb{C}$, $\alpha\beta = \beta\alpha$.
- **AACN Additive Associativity, Complex Numbers** For any $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.
- **MACN Multiplicative Associativity, Complex Numbers** For any $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.
- **DCN Distributivity, Complex Numbers** For any $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.
- **ZCN Zero, Complex Numbers** There is a complex number $0 = 0 + 0i$ so that for any $\alpha \in \mathbb{C}$, $0 + \alpha = \alpha$.
- **OCN One, Complex Numbers** There is a complex number $1 = 1 + 0i$ so that for any $\alpha \in \mathbb{C}$, $1\alpha = \alpha$.
- **AICN Additive Inverse, Complex Numbers** For every $\alpha \in \mathbb{C}$ there exists $-\alpha \in \mathbb{C}$ so that $\alpha + (-\alpha) = 0$.
- **MICN Multiplicative Inverse, Complex Numbers** For every $\alpha \in \mathbb{C}$, $\alpha \neq 0$ there exists $\frac{1}{\alpha} \in \mathbb{C}$ so that $\alpha \left(\frac{1}{\alpha}\right) = 1$.

The **conjugate** of the complex number $c = a + bi \in \mathbb{C}$ is the complex number $\bar{c} = a - bi$.

Suppose that c and d are complex numbers. Then $\overline{c + d} = \bar{c} + \bar{d}$.

Suppose that c and d are complex numbers. Then $\overline{cd} = \overline{c}\overline{d}$.

Suppose that c is a complex number. Then $\overline{\overline{c}} = c$.

The **modulus** of the complex number $c = a + bi \in \mathbb{C}$, is the nonnegative real number

$$|c| = \sqrt{c\bar{c}} = \sqrt{a^2 + b^2}.$$

A **set** is an unordered collection of objects. If S is a set and x is an object that is in the set S , we write $x \in S$. If x is not in S , then we write $x \notin S$. We refer to the objects in a set as its **elements**.

If S and T are two sets, then S is a subset of T , written $S \subseteq T$ if whenever $x \in S$ then $x \in T$.

The empty set is the set with no elements. Its is denoted by \emptyset .

Two sets, S and T , are equal, if $S \subseteq T$ and $T \subseteq S$. In this case, we write $S = T$.

Suppose S is a finite set. Then the number of elements in S is called the **cardinality** or **size** of S , and is denoted $|S|$.

Suppose S and T are sets. Then the **union** of S and T , denoted $S \cup T$, is the set whose elements are those that are elements of S or of T , or both. More formally,

$$x \in S \cup T \text{ if and only if } x \in S \text{ or } x \in T$$

Suppose S and T are sets. Then the **intersection** of S and T , denoted $S \cap T$, is the set whose elements are only those that are elements of S and of T . More formally,

$$x \in S \cap T \text{ if and only if } x \in S \text{ and } x \in T$$

Suppose S is a set that is a subset of a universal set U . Then the **complement** of S , denoted \bar{S} , is the set whose elements are those that are elements of U and not elements of S . More formally,

$$x \in \bar{S} \text{ if and only if } x \in U \text{ and } x \notin S$$

A **system of linear equations** is a collection of m equations in the variable quantities $x_1, x_2, x_3, \dots, x_n$ of the form,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where the values of a_{ij} , b_i and x_j are from the set of complex numbers, \mathbb{C} .

Two systems of linear equations are **equivalent** if their solution sets are equal.

Given a system of linear equations, the following three operations will transform the system into a different one, and each operation is known as an **equation operation**.

1. Swap the locations of two equations in the list of equations.
2. Multiply each term of an equation by a nonzero quantity.
3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

If we apply one of the three equation operations of Definition EO to a system of linear equations (Definition SLE), then the original system and the transformed system are equivalent.

An $m \times n$ **matrix** is a rectangular layout of numbers from \mathbb{C} having m rows and n columns. We will use upper-case Latin letters from the start of the alphabet (A, B, C, \dots) to denote matrices and squared-off brackets to delimit the layout. Many use large parentheses instead of brackets — the distinction is not important. Rows of a matrix will be referenced starting at the top and working down (i.e. row 1 is at the top) and columns will be referenced starting from the left (i.e. column 1 is at the left). For a matrix A , the notation $[A]_{ij}$ will refer to the complex number in row i and column j of A .

A **column vector** of **size** m is an ordered list of m numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a **vector**. Column vectors will be written in bold, usually with lower case Latin letter from the end of the alphabet such as **u**, **v**, **w**, **x**, **y**, **z**. Some books like to write vectors with arrows, such as \vec{u} . Writing by hand, some like to put arrows on top of the symbol, or a tilde underneath the symbol, as in \tilde{u} . To refer to the **entry** or **component** that is number i in the list that is the vector **v** we write $[\mathbf{v}]_i$.

The **zero vector** of size m is the column vector of size m where each entry is the number zero,

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or more compactly, $[\mathbf{0}]_i = 0$ for $1 \leq i \leq m$.

For a system of linear equations,

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\&\vdots \\a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m\end{aligned}$$

the **coefficient matrix** is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

For a system of linear equations,

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\&\vdots \\a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m\end{aligned}$$

the **vector of constants** is the column vector of size m

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

For a system of linear equations,

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\&\vdots \\a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m\end{aligned}$$

the **solution vector** is the column vector of size n

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

If A is the coefficient matrix of a system of linear equations and \mathbf{b} is the vector of constants, then we will write $\mathcal{LS}(A, \mathbf{b})$ as a shorthand expression for the system of linear equations, which we will refer to as the **matrix representation** of the linear system.

Suppose we have a system of m equations in n variables, with coefficient matrix A and vector of constants \mathbf{b} . Then the **augmented matrix** of the system of equations is the $m \times (n + 1)$ matrix whose first n columns are the columns of A and whose last column (number $n + 1$) is the column vector \mathbf{b} . This matrix will be written as $[A \mid \mathbf{b}]$.

The following three operations will transform an $m \times n$ matrix into a different matrix of the same size, and each is known as a **row operation**.

1. Swap the locations of two rows.
2. Multiply each entry of a single row by a nonzero quantity.
3. Multiply each entry of one row by some quantity, and add these values to the entries in the same columns of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

We will use a symbolic shorthand to describe these row operations:

1. $R_i \leftrightarrow R_j$: Swap the location of rows i and j .
2. αR_i : Multiply row i by the nonzero scalar α .
3. $\alpha R_i + R_j$: Multiply row i by the scalar α and add to row j .

Two matrices, A and B , are **row-equivalent** if one can be obtained from the other by a sequence of row operations.

Suppose that A and B are row-equivalent augmented matrices. Then the systems of linear equations that they represent are equivalent systems.

A matrix is in **reduced row-echelon form** if it meets all of the following conditions:

1. A row where every entry is zero lies below any row that contains a nonzero entry.
2. The leftmost nonzero entry of a row is equal to 1.
3. The leftmost nonzero entry of a row is the only nonzero entry in its column.
4. Consider any two different leftmost nonzero entries, one located in row i , column j and the other located in row s , column t . If $s > i$, then $t > j$.

A row of only zero entries will be called a **zero row** and the leftmost nonzero entry of a nonzero row will be called a **leading 1**. The number of nonzero rows will be denoted by r .

A column containing a leading 1 will be called a **pivot column**. The set of column indices for all of the pivot columns will be denoted by $D = \{d_1, d_2, d_3, \dots, d_r\}$ where $d_1 < d_2 < d_3 < \dots < d_r$, while the columns that are not pivot columns will be denoted as $F = \{f_1, f_2, f_3, \dots, f_{n-r}\}$ where $f_1 < f_2 < f_3 < \dots < f_{n-r}$.

Suppose A is a matrix. Then there is a matrix B so that

1. A and B are row-equivalent.
2. B is in reduced row-echelon form.

To **row-reduce** the matrix A means to apply row operations to A and arrive at a row-equivalent matrix B in reduced row-echelon form.

A system of linear equations is **consistent** if it has at least one solution. Otherwise, the system is called **inconsistent**.

Suppose A is the augmented matrix of a consistent system of linear equations and B is a row-equivalent matrix in reduced row-echelon form. Suppose j is the index of a column of B that contains the leading 1 for some row (i.e. column j is a pivot column), and this column is not the last column. Then the variable x_j is **dependent**. A variable that is not dependent is called **independent** or **free**.

Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then the system of equations is inconsistent if and only if the leading 1 of row r is located in column $n + 1$ of B .

Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. If $r = n + 1$, then the system of equations is inconsistent.

Suppose A is the augmented matrix of a *consistent* system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then $r \leq n$. If $r = n$, then the system has a unique solution, and if $r < n$, then the system has infinitely many solutions.

Suppose A is the augmented matrix of a *consistent* system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. Then the solution set can be described with $n - r$ free variables.

A system of linear equations has no solutions, a unique solution or infinitely many solutions.

Theorem CMVEI Consistent, More Variables than Equations, Infinite solutions
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Suppose a consistent system of linear equations has m equations in n variables. If $n > m$, then the system has infinitely many solutions.

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Definition HS Homogeneous System

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A system of linear equations, $\mathcal{LS}(A, \mathbf{b})$ is **homogeneous** if the vector of constants is the zero vector, in other words, $\mathbf{b} = \mathbf{0}$.

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Suppose that a system of linear equations is homogeneous. Then the system is consistent.

Suppose a homogeneous system of linear equations has n variables. The solution $x_1 = 0$, $x_2 = 0, \dots, x_n = 0$ (i.e. $\mathbf{x} = \mathbf{0}$) is called the **trivial solution**.

Suppose that a homogeneous system of linear equations has m equations and n variables with $n > m$. Then the system has infinitely many solutions.

The **null space** of a matrix A , denoted $\mathcal{N}(A)$, is the set of all the vectors that are solutions to the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.

A matrix with m rows and n columns is **square** if $m = n$. In this case, we say the matrix has **size** n . To emphasize the situation when a matrix is not square, we will call it **rectangular**.

Suppose A is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations $\mathcal{LS}(A, \mathbf{0})$ is $\{\mathbf{0}\}$, i.e. the system has *only* the trivial solution. Then we say that A is a **nonsingular** matrix. Otherwise we say A is a **singular** matrix.

The $m \times m$ **identity matrix**, I_m , is defined by

$$[I_m]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Suppose that A is a square matrix and B is a row-equivalent matrix in reduced row-echelon form. Then A is nonsingular if and only if B is the identity matrix.

Suppose that A is a square matrix. Then A is nonsingular if and only if the null space of A , $\mathcal{N}(A)$, contains only the zero vector, i.e. $\mathcal{N}(A) = \{\mathbf{0}\}$.

Suppose that A is a square matrix. A is a nonsingular matrix if and only if the system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every choice of the constant vector \mathbf{b} .

Suppose that A is a square matrix. The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .

The vector space \mathbb{C}^m is the set of all column vectors (Definition CV) of size m with entries from the set of complex numbers, \mathbb{C} .

Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$. Then \mathbf{u} and \mathbf{v} are **equal**, written $\mathbf{u} = \mathbf{v}$ if

$$[\mathbf{u}]_i = [\mathbf{v}]_i \quad 1 \leq i \leq m$$

Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$. The **sum** of \mathbf{u} and \mathbf{v} is the vector $\mathbf{u} + \mathbf{v}$ defined by

$$[\mathbf{u} + \mathbf{v}]_i = [\mathbf{u}]_i + [\mathbf{v}]_i \quad 1 \leq i \leq m$$

Suppose $\mathbf{u} \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$, then the **scalar multiple** of \mathbf{u} by α is the vector $\alpha\mathbf{u}$ defined by

$$[\alpha\mathbf{u}]_i = \alpha [\mathbf{u}]_i \quad 1 \leq i \leq m$$

Theorem VSPCV Vector Space Properties of Column Vectors

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Suppose that \mathbb{C}^m is the set of column vectors of size m (Definition VSCV) with addition and scalar multiplication as defined in Definition CVA and Definition CVSM. Then

- **ACC Additive Closure, Column Vectors** If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$.
- **SCC Scalar Closure, Column Vectors** If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha\mathbf{u} \in \mathbb{C}^m$.
- **CC Commutativity, Column Vectors** If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- **AAC Additive Associativity, Column Vectors** If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **ZC Zero Vector, Column Vectors** There is a vector, $\mathbf{0}$, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{C}^m$.
- **AIC Additive Inverses, Column Vectors** If $\mathbf{u} \in \mathbb{C}^m$, then there exists a vector $-\mathbf{u} \in \mathbb{C}^m$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- **SMAC Scalar Multiplication Associativity, Column Vectors** If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$.
- **DVAC Distributivity across Vector Addition, Column Vectors** If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.
- **DSAC Distributivity across Scalar Addition, Column Vectors** If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.
- **OC One, Column Vectors** If $\mathbf{u} \in \mathbb{C}^m$, then $1\mathbf{u} = \mathbf{u}$.

Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n$ from \mathbb{C}^m and n scalars $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$, their **linear combination** is the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_n \mathbf{u}_n.$$

Denote the columns of the $m \times n$ matrix A as the vectors $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n$. Then \mathbf{x} is a solution to the linear system of equations $\mathcal{LS}(A, \mathbf{b})$ if and only if

$$[\mathbf{x}]_1 \mathbf{A}_1 + [\mathbf{x}]_2 \mathbf{A}_2 + [\mathbf{x}]_3 \mathbf{A}_3 + \cdots + [\mathbf{x}]_n \mathbf{A}_n = \mathbf{b}$$

Suppose that $[A | \mathbf{b}]$ is the augmented matrix for a consistent linear system $\mathcal{LS}(A, \mathbf{b})$ of m equations in n variables. Let B be a row-equivalent $m \times (n + 1)$ matrix in reduced row-echelon form. Suppose that B has r nonzero rows, columns without leading 1's with indices $F = \{f_1, f_2, f_3, \dots, f_{n-r}, n + 1\}$, and columns with leading 1's (pivot columns) having indices $D = \{d_1, d_2, d_3, \dots, d_r\}$. Define vectors $\mathbf{c}, \mathbf{u}_j, 1 \leq j \leq n - r$ of size n by

$$[\mathbf{c}]_i = \begin{cases} 0 & \text{if } i \in F \\ [B]_{k,n+1} & \text{if } i \in D, i = d_k \end{cases}$$

$$[\mathbf{u}_j]_i = \begin{cases} 1 & \text{if } i \in F, i = f_j \\ 0 & \text{if } i \in F, i \neq f_j \\ -[B]_{k,f_j} & \text{if } i \in D, i = d_k \end{cases}.$$

Then the set of solutions to the system of equations $\mathcal{LS}(A, \mathbf{b})$ is

$$S = \left\{ \mathbf{c} + x_{f_1} \mathbf{u}_1 + x_{f_2} \mathbf{u}_2 + x_{f_3} \mathbf{u}_3 + \cdots + x_{f_{n-r}} \mathbf{u}_{n-r} \mid x_{f_1}, x_{f_2}, x_{f_3}, \dots, x_{f_{n-r}} \in \mathbb{C} \right\}$$

Suppose that \mathbf{w} is one solution to the linear system of equations $\mathcal{LS}(A, b)$. Then \mathbf{y} is a solution to $\mathcal{LS}(A, b)$ if and only if $\mathbf{y} = \mathbf{w} + \mathbf{z}$ for some vector $\mathbf{z} \in \mathcal{N}(A)$.

Suppose that A is an $m \times n$ matrix and that B and C are $m \times n$ matrices that are row-equivalent to A and in reduced row-echelon form. Then $B = C$.

Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p\}$, their **span**, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p$. Symbolically,

$$\begin{aligned}\langle S \rangle &= \{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_p \mathbf{u}_p \mid \alpha_i \in \mathbb{C}, 1 \leq i \leq p \} \\ &= \left\{ \sum_{i=1}^p \alpha_i \mathbf{u}_i \mid \alpha_i \in \mathbb{C}, 1 \leq i \leq p \right\}\end{aligned}$$

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \dots, d_r\}$ be the column indices where B has leading 1's (pivot columns) and $F = \{f_1, f_2, f_3, \dots, f_{n-r}\}$ be the set of column indices where B does not have leading 1's. Construct the $n - r$ vectors \mathbf{z}_j , $1 \leq j \leq n - r$ of size n as

$$[\mathbf{z}_j]_i = \begin{cases} 1 & \text{if } i \in F, i = f_j \\ 0 & \text{if } i \in F, i \neq f_j \\ -[B]_{k,f_j} & \text{if } i \in D, i = d_k \end{cases}$$

Then the null space of A is given by

$$\mathcal{N}(A) = \langle \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \dots, \mathbf{z}_{n-r}\} \rangle.$$

Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$, a true statement of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

is a **relation of linear dependence** on S . If this statement is formed in a trivial fashion, i.e. $\alpha_i = 0$, $1 \leq i \leq n$, then we say it is the **trivial relation of linear dependence** on S .

The set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$ is the set of vectors in \mathbb{C}^m that are the columns of A . Then S is a linearly independent set if and only if the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has a unique solution.

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$ is the set of vectors in \mathbb{C}^m that are the columns of A . Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of non-zero rows in B . Then S is linearly independent if and only if $n = r$.

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is the set of vectors in \mathbb{C}^m , and that $n > m$. Then S is a linearly dependent set.

Theorem NMLIC Nonsingular Matrices have Linearly Independent Columns 73

Suppose that A is a square matrix. Then A is nonsingular if and only if the columns of A form a linearly independent set.

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Theorem NME2 Nonsingular Matrix Equivalences, Round 2

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Suppose that A is a square matrix. The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
5. The columns of A form a linearly independent set.

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Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \dots, d_r\}$ and $F = \{f_1, f_2, f_3, \dots, f_{n-r}\}$ be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the $n - r$ vectors \mathbf{z}_j , $1 \leq j \leq n - r$ of size n as

$$[\mathbf{z}_j]_i = \begin{cases} 1 & \text{if } i \in F, i = f_j \\ 0 & \text{if } i \in F, i \neq f_j \\ -[B]_{k,f_j} & \text{if } i \in D, i = d_k \end{cases}$$

Define the set $S = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \dots, \mathbf{z}_{n-r}\}$. Then

1. $\mathcal{N}(A) = \langle S \rangle$.
2. S is a linearly independent set.

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a set of vectors. Then S is a linearly dependent set if and only if there is an index t , $1 \leq t \leq n$ such that \mathbf{u}_t is a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_{t-1}, \mathbf{u}_{t+1}, \dots, \mathbf{u}_n$.

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$ is a set of column vectors. Define $W = \langle S \rangle$ and let A be the matrix whose columns are the vectors from S . Let B be the reduced row-echelon form of A , with $D = \{d_1, d_2, d_3, \dots, d_r\}$ the set of column indices corresponding to the pivot columns of B . Then

1. $T = \{\mathbf{v}_{d_1}, \mathbf{v}_{d_2}, \mathbf{v}_{d_3}, \dots, \mathbf{v}_{d_r}\}$ is a linearly independent set.
2. $W = \langle T \rangle$.

Suppose that \mathbf{u} is a vector from \mathbb{C}^m . Then the conjugate of the vector, $\bar{\mathbf{u}}$, is defined by

$$[\bar{\mathbf{u}}]_i = \overline{[\mathbf{u}]_i} \quad 1 \leq i \leq m$$

Suppose \mathbf{x} and \mathbf{y} are two vectors from \mathbb{C}^m . Then

$$\overline{\mathbf{x} + \mathbf{y}} = \overline{\mathbf{x}} + \overline{\mathbf{y}}$$

Suppose \mathbf{x} is a vector from \mathbb{C}^m , and $\alpha \in \mathbb{C}$ is a scalar. Then

$$\overline{\alpha \mathbf{x}} = \overline{\alpha} \overline{\mathbf{x}}$$

Given the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ the **inner product** of \mathbf{u} and \mathbf{v} is the scalar quantity in \mathbb{C} ,

$$\langle \mathbf{u}, \mathbf{v} \rangle = [\mathbf{u}]_1 \overline{[\mathbf{v}]_1} + [\mathbf{u}]_2 \overline{[\mathbf{v}]_2} + [\mathbf{u}]_3 \overline{[\mathbf{v}]_3} + \cdots + [\mathbf{u}]_m \overline{[\mathbf{v}]_m} = \sum_{i=1}^m [\mathbf{u}]_i \overline{[\mathbf{v}]_i}$$

Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$. Then

1. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$. Then

1. $\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$
2. $\langle \mathbf{u}, \alpha \mathbf{v} \rangle = \bar{\alpha} \langle \mathbf{u}, \mathbf{v} \rangle$

Suppose that \mathbf{u} and \mathbf{v} are vectors in \mathbb{C}^m . Then $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$.

The **norm** of the vector \mathbf{u} is the scalar quantity in \mathbb{C}

$$\|\mathbf{u}\| = \sqrt{|[\mathbf{u}]_1|^2 + |[\mathbf{u}]_2|^2 + |[\mathbf{u}]_3|^2 + \cdots + |[\mathbf{u}]_m|^2} = \sqrt{\sum_{i=1}^m |[\mathbf{u}]_i|^2}$$

Suppose that \mathbf{u} is a vector in \mathbb{C}^m . Then $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle$.

Suppose that \mathbf{u} is a vector in \mathbb{C}^m . Then $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$.

A pair of vectors, \mathbf{u} and \mathbf{v} , from \mathbb{C}^m are **orthogonal** if their inner product is zero, that is, $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a set of vectors from \mathbb{C}^m . Then S is an **orthogonal set** if every pair of different vectors from S is orthogonal, that is $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ whenever $i \neq j$.

Let $\mathbf{e}_j \in \mathbb{C}^m$, $1 \leq j \leq m$ denote the column vectors defined by

$$[\mathbf{e}_j]_i = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Then the set

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_m\} = \{\mathbf{e}_j \mid 1 \leq j \leq m\}$$

is the set of **standard unit vectors** in \mathbb{C}^m .

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is an orthogonal set of nonzero vectors. Then S is linearly independent.

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p\}$ is a linearly independent set of vectors in \mathbb{C}^m . Define the vectors \mathbf{u}_i , $1 \leq i \leq p$ by

$$\mathbf{u}_i = \mathbf{v}_i - \frac{\langle \mathbf{v}_i, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{v}_i, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \mathbf{u}_2 - \frac{\langle \mathbf{v}_i, \mathbf{u}_3 \rangle}{\langle \mathbf{u}_3, \mathbf{u}_3 \rangle} \mathbf{u}_3 - \dots - \frac{\langle \mathbf{v}_i, \mathbf{u}_{i-1} \rangle}{\langle \mathbf{u}_{i-1}, \mathbf{u}_{i-1} \rangle} \mathbf{u}_{i-1}$$

Then if $T = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p\}$, then T is an orthogonal set of non-zero vectors, and $\langle T \rangle = \langle S \rangle$.

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is an orthogonal set of vectors such that $\|\mathbf{u}_i\| = 1$ for all $1 \leq i \leq n$. Then S is an **orthonormal** set of vectors.

The vector space M_{mn} is the set of all $m \times n$ matrices with entries from the set of complex numbers.

The $m \times n$ matrices A and B are **equal**, written $A = B$ provided $[A]_{ij} = [B]_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$.

Given the $m \times n$ matrices A and B , define the **sum** of A and B as an $m \times n$ matrix, written $A + B$, according to

$$[A + B]_{ij} = [A]_{ij} + [B]_{ij} \quad 1 \leq i \leq m, 1 \leq j \leq n$$

Given the $m \times n$ matrix A and the scalar $\alpha \in \mathbb{C}$, the **scalar multiple** of A is an $m \times n$ matrix, written αA and defined according to

$$[\alpha A]_{ij} = \alpha [A]_{ij} \quad 1 \leq i \leq m, 1 \leq j \leq n$$

Theorem VSPM Vector Space Properties of Matrices

Suppose that M_{mn} is the set of all $m \times n$ matrices (Definition VSM) with addition and scalar multiplication as defined in Definition MA and Definition MSM. Then

- **ACM Additive Closure, Matrices** If $A, B \in M_{mn}$, then $A + B \in M_{mn}$.
- **SCM Scalar Closure, Matrices** If $\alpha \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha A \in M_{mn}$.
- **CM Commutativity, Matrices** If $A, B \in M_{mn}$, then $A + B = B + A$.
- **AAM Additive Associativity, Matrices** If $A, B, C \in M_{mn}$, then $A + (B + C) = (A + B) + C$.
- **ZM Zero Vector, Matrices** There is a matrix, \mathcal{O} , called the **zero matrix**, such that $A + \mathcal{O} = A$ for all $A \in M_{mn}$.
- **AIM Additive Inverses, Matrices** If $A \in M_{mn}$, then there exists a matrix $-A \in M_{mn}$ so that $A + (-A) = \mathcal{O}$.
- **SMAM Scalar Multiplication Associativity, Matrices** If $\alpha, \beta \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha(\beta A) = (\alpha\beta)A$.
- **DMAM Distributivity across Matrix Addition, Matrices** If $\alpha \in \mathbb{C}$ and $A, B \in M_{mn}$, then $\alpha(A + B) = \alpha A + \alpha B$.
- **DSAM Distributivity across Scalar Addition, Matrices** If $\alpha, \beta \in \mathbb{C}$ and $A \in M_{mn}$, then $(\alpha + \beta)A = \alpha A + \beta A$.
- **OM One, Matrices** If $A \in M_{mn}$, then $1A = A$.

The $m \times n$ **zero matrix** is written as $\mathcal{O} = \mathcal{O}_{m \times n}$ and defined by $[\mathcal{O}]_{ij} = 0$, for all $1 \leq i \leq m$, $1 \leq j \leq n$.

Given an $m \times n$ matrix A , its **transpose** is the $n \times m$ matrix A^t given by

$$[A^t]_{ij} = [A]_{ji}, \quad 1 \leq i \leq n, 1 \leq j \leq m.$$

The matrix A is **symmetric** if $A = A^t$.

Suppose that A is a symmetric matrix. Then A is square.

Suppose that A and B are $m \times n$ matrices. Then $(A + B)^t = A^t + B^t$.

Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $(\alpha A)^t = \alpha A^t$.

Suppose that A is an $m \times n$ matrix. Then $(A^t)^t = A$.

Suppose A is an $m \times n$ matrix. Then the **conjugate** of A , written \overline{A} is an $m \times n$ matrix defined by

$$[A]_{ij} = \overline{[\overline{A}]_{ij}}$$

Suppose that A and B are $m \times n$ matrices. Then $\overline{A + B} = \overline{A} + \overline{B}$.

Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $\overline{\alpha A} = \overline{\alpha} \overline{A}$.

Suppose that A is an $m \times n$ matrix. Then $\overline{\overline{A}} = A$.

Suppose that A is an $m \times n$ matrix. Then $\overline{A^t} = (\overline{A})^t$.

If A is a square matrix, then its **adjoint** is $A^* = (\overline{A})^t$.

Suppose A and B are matrices of the same size. Then $(A + B)^* = A^* + B^*$.

Suppose $\alpha \in \mathbb{C}$ is a scalar and A is a matrix. Then $(\alpha A)^* = \bar{\alpha}A^*$.

Suppose that A is a matrix. Then $(A^*)^* = A$

Suppose A is an $m \times n$ matrix with columns $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n$ and \mathbf{u} is a vector of size n . Then the **matrix-vector product** of A with \mathbf{u} is the linear combination

$$A\mathbf{u} = [\mathbf{u}]_1 \mathbf{A}_1 + [\mathbf{u}]_2 \mathbf{A}_2 + [\mathbf{u}]_3 \mathbf{A}_3 + \cdots + [\mathbf{u}]_n \mathbf{A}_n$$

The set of solutions to the linear system $\mathcal{LS}(A, \mathbf{b})$ equals the set of solutions for \mathbf{x} in the vector equation $A\mathbf{x} = \mathbf{b}$.

Suppose that A and B are $m \times n$ matrices such that $A\mathbf{x} = B\mathbf{x}$ for every $\mathbf{x} \in \mathbb{C}^n$. Then $A = B$.

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \dots, \mathbf{B}_p$. Then the **matrix product** of A with B is the $m \times p$ matrix where column i is the matrix-vector product $A\mathbf{B}_i$. Symbolically,

$$AB = A[\mathbf{B}_1|\mathbf{B}_2|\mathbf{B}_3|\dots|\mathbf{B}_p] = [A\mathbf{B}_1|A\mathbf{B}_2|A\mathbf{B}_3|\dots|A\mathbf{B}_p].$$

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then for $1 \leq i \leq m$, $1 \leq j \leq p$, the individual entries of AB are given by

$$\begin{aligned} [AB]_{ij} &= [A]_{i1} [B]_{1j} + [A]_{i2} [B]_{2j} + [A]_{i3} [B]_{3j} + \cdots + [A]_{in} [B]_{nj} \\ &= \sum_{k=1}^n [A]_{ik} [B]_{kj} \end{aligned}$$

Suppose A is an $m \times n$ matrix. Then

1. $A\mathcal{O}_{n \times p} = \mathcal{O}_{m \times p}$
2. $\mathcal{O}_{p \times m}A = \mathcal{O}_{p \times n}$

Suppose A is an $m \times n$ matrix. Then

1. $AI_n = A$
2. $I_m A = A$

Suppose A is an $m \times n$ matrix and B and C are $n \times p$ matrices and D is a $p \times s$ matrix. Then

1. $A(B + C) = AB + AC$
2. $(B + C)D = BD + CD$

Theorem MMSMM Matrix Multiplication and Scalar Matrix Multiplication 123

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Let α be a scalar. Then $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

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Theorem MMA Matrix Multiplication is Associative

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Suppose A is an $m \times n$ matrix, B is an $n \times p$ matrix and D is a $p \times s$ matrix. Then $A(BD) = (AB)D$.

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If we consider the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ as $m \times 1$ matrices then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^t \bar{\mathbf{v}}$$

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $\overline{AB} = \bar{A} \bar{B}$.

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $(AB)^t = B^t A^t$.

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $(AB)^* = B^* A^*$.

Suppose that A is an $m \times n$ matrix and $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{y} \in \mathbb{C}^m$. Then $\langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A^*\mathbf{y} \rangle$.

The square matrix A is **Hermitian** (or **self-adjoint**) if $A = A^*$.

Suppose that A is a square matrix of size n . Then A is Hermitian if and only if $\langle Ax, y \rangle = \langle x, Ay \rangle$ for all $x, y \in \mathbb{C}^n$.

Suppose A and B are square matrices of size n such that $AB = I_n$ and $BA = I_n$. Then A is **invertible** and B is the **inverse** of A . In this situation, we write $B = A^{-1}$.

Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is invertible if and only if $ad - bc \neq 0$. When A is invertible, we have

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Suppose A is a nonsingular square matrix of size n . Create the $n \times 2n$ matrix M by placing the $n \times n$ identity matrix I_n to the right of the matrix A . Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Finally, let J be the matrix formed from the final n columns of N . Then $AJ = I_n$.

Suppose the square matrix A has an inverse. Then A^{-1} is unique.

Suppose A and B are invertible matrices of size n . Then $(AB)^{-1} = B^{-1}A^{-1}$ and AB is an invertible matrix.

Suppose A is an invertible matrix. Then A^{-1} is invertible and $(A^{-1})^{-1} = A$.

Suppose A is an invertible matrix. Then A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

Suppose A is an invertible matrix and α is a nonzero scalar. Then $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$ and αA is invertible.

Suppose that A and B are square matrices of size n and the product AB is nonsingular. Then A and B are both nonsingular.

Suppose A and B are square matrices of size n such that $AB = I_n$. Then $BA = I_n$.

Suppose that A is a square matrix. Then A is nonsingular if and only if A is invertible.

Suppose that A is a square matrix of size n . The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
5. The columns of A are a linearly independent set.
6. A is invertible.

Suppose that A is nonsingular. Then the unique solution to $\mathcal{LS}(A, \mathbf{b})$ is $A^{-1}\mathbf{b}$.

Suppose that U is a square matrix of size n such that $U^*U = I_n$. Then we say U is **unitary**.

Suppose that U is a unitary matrix of size n . Then U is nonsingular, and $U^{-1} = U^*$.

Suppose that A is a square matrix of size n with columns $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$. Then A is a unitary matrix if and only if S is an orthonormal set.

Suppose that U is a unitary matrix of size n and \mathbf{u} and \mathbf{v} are two vectors from \mathbb{C}^n . Then

$$\langle U\mathbf{u}, U\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle \quad \text{and} \quad \|U\mathbf{v}\| = \|\mathbf{v}\|$$

Suppose that A is an $m \times n$ matrix with columns $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$. Then the **column space** of A , written $\mathcal{C}(A)$, is the subset of \mathbb{C}^m containing all linear combinations of the columns of A ,

$$\mathcal{C}(A) = \langle \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\} \rangle$$

Suppose A is an $m \times n$ matrix and \mathbf{b} is a vector of size m . Then $\mathbf{b} \in \mathcal{C}(A)$ if and only if $\mathcal{LS}(A, \mathbf{b})$ is consistent.

Suppose that A is an $m \times n$ matrix with columns $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n$, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \dots, d_r\}$ be the set of column indices where B has leading 1's. Let $T = \{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}, \mathbf{A}_{d_3}, \dots, \mathbf{A}_{d_r}\}$. Then

1. T is a linearly independent set.
2. $\mathcal{C}(A) = \langle T \rangle$.

Suppose A is a square matrix of size n . Then A is nonsingular if and only if $\mathcal{C}(A) = \mathbb{C}^n$.

Suppose that A is a square matrix of size n . The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
5. The columns of A are a linearly independent set.
6. A is invertible.
7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.

Suppose A is an $m \times n$ matrix. Then the **row space** of A , $\mathcal{R}(A)$, is the column space of A^t , i.e. $\mathcal{R}(A) = \mathcal{C}(A^t)$.

Suppose A and B are row-equivalent matrices. Then $\mathcal{R}(A) = \mathcal{R}(B)$.

Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form. Let S be the set of nonzero columns of B^t . Then

1. $\mathcal{R}(A) = \langle S \rangle$.
2. S is a linearly independent set.

Suppose A is a matrix. Then $\mathcal{C}(A) = \mathcal{R}(A^t)$.

Suppose A is an $m \times n$ matrix. Then the **left null space** is defined as $\mathcal{L}(A) = \mathcal{N}(A^t) \subseteq \mathbb{C}^m$.

Suppose A is an $m \times n$ matrix. Add m new columns to A that together equal an $m \times m$ identity matrix to form an $m \times (n+m)$ matrix M . Use row operations to bring M to reduced row-echelon form and call the result N . N is the **extended reduced row-echelon form** of A , and we will standardize on names for five submatrices (B, C, J, K, L) of N .

Let B denote the $m \times n$ matrix formed from the first n columns of N and let J denote the $m \times m$ matrix formed from the last m columns of N . Suppose that B has r nonzero rows. Further partition N by letting C denote the $r \times n$ matrix formed from all of the non-zero rows of B . Let K be the $r \times m$ matrix formed from the first r rows of J , while L will be the $(m-r) \times m$ matrix formed from the bottom $m-r$ rows of J . Pictorially,

$$M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \left[\begin{array}{c|c} C & K \\ \hline 0 & L \end{array} \right]$$

Suppose that A is an $m \times n$ matrix and that N is its extended echelon form. Then

1. J is nonsingular.
2. $B = JA$.
3. If $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$, then $A\mathbf{x} = \mathbf{y}$ if and only if $B\mathbf{x} = J\mathbf{y}$.
4. C is in reduced row-echelon form, has no zero rows and has r pivot columns.
5. L is in reduced row-echelon form, has no zero rows and has $m-r$ pivot columns.

Suppose A is an $m \times n$ matrix with extended echelon form N . Suppose the reduced row-echelon form of A has r nonzero rows. Then C is the submatrix of N formed from the first r rows and the first n columns and L is the submatrix of N formed from the last m columns and the last $m - r$ rows. Then

1. The null space of A is the null space of C , $\mathcal{N}(A) = \mathcal{N}(C)$.
2. The row space of A is the row space of C , $\mathcal{R}(A) = \mathcal{R}(C)$.
3. The column space of A is the null space of L , $\mathcal{C}(A) = \mathcal{N}(L)$.
4. The left null space of A is the row space of L , $\mathcal{L}(A) = \mathcal{R}(L)$.

Definition VS Vector Space

Suppose that V is a set upon which we have defined two operations: (1) **vector addition**, which combines two elements of V and is denoted by “+”, and (2) **scalar multiplication**, which combines a complex number with an element of V and is denoted by juxtaposition. Then V , along with the two operations, is a **vector space** if the following ten properties hold.

- **AC Additive Closure** If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.
- **SC Scalar Closure** If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha\mathbf{u} \in V$.
- **C Commutativity** If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- **AA Additive Associativity** If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- **Z Zero Vector** There is a vector, $\mathbf{0}$, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$.
- **AI Additive Inverses** If $\mathbf{u} \in V$, then there exists a vector $-\mathbf{u} \in V$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- **SMA Scalar Multiplication Associativity** If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$.
- **DVA Distributivity across Vector Addition** If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in V$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.
- **DSA Distributivity across Scalar Addition** If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.
- **O One** If $\mathbf{u} \in V$, then $1\mathbf{u} = \mathbf{u}$.

The objects in V are called **vectors**, no matter what else they might really be, simply by virtue of being elements of a vector space.

Suppose that V is a vector space. The zero vector, $\mathbf{0}$, is unique.

Suppose that V is a vector space. For each $\mathbf{u} \in V$, the additive inverse, $-\mathbf{u}$, is unique.

Suppose that V is a vector space and $\mathbf{u} \in V$. Then $0\mathbf{u} = \mathbf{0}$.

Suppose that V is a vector space and $\alpha \in \mathbb{C}$. Then $\alpha\mathbf{0} = \mathbf{0}$.

Suppose that V is a vector space and $\mathbf{u} \in V$. Then $-\mathbf{u} = (-1)\mathbf{u}$.

Suppose that V is a vector space and $\alpha \in \mathbb{C}$. If $\alpha\mathbf{u} = \mathbf{0}$, then either $\alpha = 0$ or $\mathbf{u} = \mathbf{0}$.

Suppose that V and W are two vector spaces that have identical definitions of vector addition and scalar multiplication, and that W is a subset of V , $W \subseteq V$. Then W is a **subspace** of V .

Suppose that V is a vector space and W is a subset of V , $W \subseteq V$. Endow W with the same operations as V . Then W is a subspace if and only if three conditions are met

1. W is non-empty, $W \neq \emptyset$.
2. If $\mathbf{x} \in W$ and $\mathbf{y} \in W$, then $\mathbf{x} + \mathbf{y} \in W$.
3. If $\alpha \in \mathbb{C}$ and $\mathbf{x} \in W$, then $\alpha\mathbf{x} \in W$.

Given the vector space V , the subspaces V and $\{\mathbf{0}\}$ are each called a **trivial subspace**.

Suppose that A is an $m \times n$ matrix. Then the null space of A , $\mathcal{N}(A)$, is a subspace of \mathbb{C}^n .

Suppose that V is a vector space. Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n$ and n scalars $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$, their **linear combination** is the vector

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n.$$

Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$, their **span**, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t$. Symbolically,

$$\begin{aligned} \langle S \rangle &= \{ \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_t\mathbf{u}_t \mid \alpha_i \in \mathbb{C}, 1 \leq i \leq t \} \\ &= \left\{ \sum_{i=1}^t \alpha_i\mathbf{u}_i \mid \alpha_i \in \mathbb{C}, 1 \leq i \leq t \right\} \end{aligned}$$

Suppose V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\} \subseteq V$, their span, $\langle S \rangle$, is a subspace.

Suppose that A is an $m \times n$ matrix. Then $\mathcal{C}(A)$ is a subspace of \mathbb{C}^m .

Suppose that A is an $m \times n$ matrix. Then $\mathcal{R}(A)$ is a subspace of \mathbb{C}^n .

Suppose that A is an $m \times n$ matrix. Then $\mathcal{L}(A)$ is a subspace of \mathbb{C}^m .

Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$, an equation of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

is a **relation of linear dependence** on S . If this equation is formed in a trivial fashion, i.e. $\alpha_i = 0, 1 \leq i \leq n$, then we say it is a **trivial relation of linear dependence** on S .

Suppose that V is a vector space. The set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ from V is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

Suppose V is a vector space. A subset S of V is a **spanning set** for V if $\langle S \rangle = V$. In this case, we also say S **spans** V .

Suppose that V is a vector space and $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m\}$ is a linearly independent set that spans V . Let \mathbf{w} be any vector in V . Then there exist *unique* scalars $a_1, a_2, a_3, \dots, a_m$ such that

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_m\mathbf{v}_m.$$

Suppose V is a vector space. Then a subset $S \subseteq V$ is a **basis** of V if it is linearly independent and spans V .

The set of standard unit vectors for \mathbb{C}^m (Definition SUV), $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_m\} = \{\mathbf{e}_i \mid 1 \leq i \leq m\}$ is a basis for the vector space \mathbb{C}^m .

Suppose that A is a square matrix of size m . Then the columns of A are a basis of \mathbb{C}^m if and only if A is nonsingular.

Suppose that A is a square matrix of size n . The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
5. The columns of A are a linearly independent set.
6. A is invertible.
7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
8. The columns of A are a basis for \mathbb{C}^n .

Suppose that $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p\}$ is an orthonormal basis of the subspace W of \mathbb{C}^m . For any $\mathbf{w} \in W$,

$$\mathbf{w} = \langle \mathbf{w}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{w}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \langle \mathbf{w}, \mathbf{v}_3 \rangle \mathbf{v}_3 + \cdots + \langle \mathbf{w}, \mathbf{v}_p \rangle \mathbf{v}_p$$

Suppose that U is a unitary matrix of size n and $B = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n\}$ is an orthonormal basis of \mathbb{C}^n . Then the set $C = \{U\mathbf{x}_1, U\mathbf{x}_2, U\mathbf{x}_3, \dots, U\mathbf{x}_n\}$ is an orthonormal basis of \mathbb{C}^n .

Suppose that V is a vector space and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t\}$ is a basis of V . Then the **dimension** of V is defined by $\dim(V) = t$. If V has no finite bases, we say V has infinite dimension.

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t\}$ is a finite set of vectors which spans the vector space V . Then any set of $t + 1$ or more vectors from V is linearly dependent.

Suppose that V is a vector space with a finite basis B and a second basis C . Then B and C have the same size.

The dimension of \mathbb{C}^m (Example VSCV) is m .

The dimension of P_n (Example VSP) is $n + 1$.

The dimension of M_{mn} (Example VSM) is mn .

Suppose that A is an $m \times n$ matrix. Then the **nullity** of A is the dimension of the null space of A , $n(A) = \dim(\mathcal{N}(A))$.

Suppose that A is an $m \times n$ matrix. Then the **rank** of A is the dimension of the column space of A , $r(A) = \dim(\mathcal{C}(A))$.

Suppose that A is an $m \times n$ matrix and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then $r(A) = r$ and $n(A) = n - r$.

Suppose that A is an $m \times n$ matrix. Then $r(A) + n(A) = n$.

Suppose that A is a square matrix of size n . The following are equivalent.

1. A is nonsingular.
2. The rank of A is n , $r(A) = n$.
3. The nullity of A is zero, $n(A) = 0$.

Suppose that A is a square matrix of size n . The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
5. The columns of A are a linearly independent set.
6. A is invertible.
7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
8. The columns of A are a basis for \mathbb{C}^n .
9. The rank of A is n , $r(A) = n$.
10. The nullity of A is zero, $n(A) = 0$.

Suppose V is vector space and S is a linearly independent set of vectors from V . Suppose \mathbf{w} is a vector such that $\mathbf{w} \notin \langle S \rangle$. Then the set $S' = S \cup \{\mathbf{w}\}$ is linearly independent.

Suppose that V is a vector space of dimension t . Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m\}$ be a set of vectors from V . Then

1. If $m > t$, then S is linearly dependent.
2. If $m < t$, then S does not span V .
3. If $m = t$ and S is linearly independent, then S spans V .
4. If $m = t$ and S spans V , then S is linearly independent.

Suppose that U and V are subspaces of the vector space W , such that $U \subsetneq V$. Then $\dim(U) < \dim(V)$.

Suppose that U and V are subspaces of the vector space W , such that $U \subseteq V$ and $\dim(U) = \dim(V)$. Then $U = V$.

Suppose A is an $m \times n$ matrix. Then $r(A) = r(A^t)$.

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then

1. $\dim(\mathcal{N}(A)) = n - r$
2. $\dim(\mathcal{C}(A)) = r$
3. $\dim(\mathcal{R}(A)) = r$
4. $\dim(\mathcal{L}(A)) = m - r$

Suppose that V is a vector space with two subspaces U and W such that for every $\mathbf{v} \in V$,

1. There exists vectors $\mathbf{u} \in U$, $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$
2. If $\mathbf{v} = \mathbf{u}_1 + \mathbf{w}_1$ and $\mathbf{v} = \mathbf{u}_2 + \mathbf{w}_2$ where $\mathbf{u}_1, \mathbf{u}_2 \in U$, $\mathbf{w}_1, \mathbf{w}_2 \in W$ then $\mathbf{u}_1 = \mathbf{u}_2$ and $\mathbf{w}_1 = \mathbf{w}_2$.

Then V is the **direct sum** of U and W and we write $V = U \oplus W$.

Suppose that V is a vector space with a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$. Define

$$U = \langle \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m\} \rangle \quad W = \langle \{\mathbf{v}_{m+1}, \mathbf{v}_{m+2}, \mathbf{v}_{m+3}, \dots, \mathbf{v}_n\} \rangle$$

Then $V = U \oplus W$.

Suppose that U is a subspace of the vector space V . Then there exists a subspace W of V such that $V = U \oplus W$.

Suppose U and W are subspaces of the vector space V . Then $V = U \oplus W$ if and only if

1. For every $\mathbf{v} \in V$, there exists vectors $\mathbf{u} \in U$, $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$.
2. Whenever $\mathbf{0} = \mathbf{u} + \mathbf{w}$ with $\mathbf{u} \in U$, $\mathbf{w} \in W$ then $\mathbf{u} = \mathbf{w} = \mathbf{0}$.

Suppose U and W are subspaces of the vector space V . Then $V = U \oplus W$ if and only if

1. For every $\mathbf{v} \in V$, there exists vectors $\mathbf{u} \in U$, $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$.
2. $U \cap W = \{\mathbf{0}\}$.

Suppose U and W are subspaces of the vector space V with $V = U \oplus W$. Suppose that R is a linearly independent subset of U and S is a linearly independent subset of W . Then $R \cup S$ is a linearly independent subset of V .

Suppose U and W are subspaces of the vector space V with $V = U \oplus W$. Then $\dim(V) = \dim(U) + \dim(W)$.

Suppose V is a vector space with subspaces U and W with $V = U \oplus W$. Suppose that X and Y are subspaces of W with $W = X \oplus Y$. Then $V = U \oplus X \oplus Y$.

1. $E_{i,j}$ is the square matrix of size n with

$$[E_{i,j}]_{k\ell} = \begin{cases} 0 & k \neq i, k \neq j, \ell \neq k \\ 1 & k \neq i, k \neq j, \ell = k \\ 0 & k = i, \ell \neq j \\ 1 & k = i, \ell = j \\ 0 & k = j, \ell \neq i \\ 1 & k = j, \ell = i \end{cases}$$

2. $E_i(\alpha)$, for $\alpha \neq 0$, is the square matrix of size n with

$$[E_i(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq i, \ell \neq k \\ 1 & k \neq i, \ell = k \\ \alpha & k = i, \ell = i \end{cases}$$

3. $E_{i,j}(\alpha)$ is the square matrix of size n with

$$[E_{i,j}(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq j, \ell \neq k \\ 1 & k \neq j, \ell = k \\ 0 & k = j, \ell \neq i, \ell \neq j \\ 1 & k = j, \ell = j \\ \alpha & k = j, \ell = i \end{cases}$$

Suppose that A is an $m \times n$ matrix, and B is a matrix of the same size that is obtained from A by a single row operation (Definition RO). Then there is an elementary matrix of size m that will convert A to B via matrix multiplication on the left. More precisely,

1. If the row operation swaps rows i and j , then $B = E_{i,j}A$.
2. If the row operation multiplies row i by α , then $B = E_i(\alpha)A$.
3. If the row operation multiplies row i by α and adds the result to row j , then $B = E_{i,j}(\alpha)A$.

If E is an elementary matrix, then E is nonsingular.

Theorem NMPM Nonsingular Matrices are Products of Elementary Matrices
218

Suppose that A is a nonsingular matrix. Then there exists elementary matrices $E_1, E_2, E_3, \dots, E_t$ so that $A = E_1 E_2 E_3 \dots E_t$.

Suppose that A is an $m \times n$ matrix. Then the **submatrix** $A(i|j)$ is the $(m-1) \times (n-1)$ matrix obtained from A by removing row i and column j .

Suppose A is a square matrix. Then its **determinant**, $\det(A) = |A|$, is an element of \mathbb{C} defined recursively by:

If A is a 1×1 matrix, then $\det(A) = [A]_{11}$.

If A is a matrix of size n with $n \geq 2$, then

$$\det(A) = [A]_{11} \det(A(1|1)) - [A]_{12} \det(A(1|2)) + [A]_{13} \det(A(1|3)) - [A]_{14} \det(A(1|4)) + \cdots + (-1)^{n+1} [A]_{1n} \det(A(1|n))$$

Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\det(A) = ad - bc$

Suppose that A is a square matrix of size n . Then

$$\det(A) = (-1)^{i+1} [A]_{i1} \det(A(i|1)) + (-1)^{i+2} [A]_{i2} \det(A(i|2)) \\ + (-1)^{i+3} [A]_{i3} \det(A(i|3)) + \cdots + (-1)^{i+n} [A]_{in} \det(A(i|n)) \quad 1 \leq i \leq n$$

which is known as **expansion** about row i .

Suppose that A is a square matrix. Then $\det(A^t) = \det(A)$.

Suppose that A is a square matrix of size n . Then

$$\begin{aligned} \det(A) = & (-1)^{1+j} [A]_{1j} \det(A(1|j)) + (-1)^{2+j} [A]_{2j} \det(A(2|j)) \\ & + (-1)^{3+j} [A]_{3j} \det(A(3|j)) + \cdots + (-1)^{n+j} [A]_{nj} \det(A(n|j)) \quad 1 \leq j \leq n \end{aligned}$$

which is known as **expansion** about column j .

Suppose that A is a square matrix with a row where every entry is zero, or a column where every entry is zero. Then $\det(A) = 0$.

Suppose that A is a square matrix. Let B be the square matrix obtained from A by interchanging the location of two rows, or interchanging the location of two columns. Then $\det(B) = -\det(A)$.

Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a single row by the scalar α , or by multiplying a single column by the scalar α . Then $\det(B) = \alpha \det(A)$.

Suppose that A is a square matrix with two equal rows, or two equal columns. Then $\det(A) = 0$.

Theorem DRCMA Determinant for Row or Column Multiples and Addition 229

Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a row by the scalar α and then adding it to another row, or by multiplying a column by the scalar α and then adding it to another column. Then $\det(B) = \det(A)$.

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Theorem DIM Determinant of the Identity Matrix

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For every $n \geq 1$, $\det(I_n) = 1$.

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For the three possible versions of an elementary matrix (Definition ELEM) we have the determinants,

1. $\det(E_{i,j}) = -1$
2. $\det(E_i(\alpha)) = \alpha$
3. $\det(E_{i,j}(\alpha)) = 1$

Theorem DEMMM Determinants, Elementary Matrices, Matrix Multiplication
232

Suppose that A is a square matrix of size n and E is any elementary matrix of size n . Then

$$\det(EA) = \det(E) \det(A)$$

Let A be a square matrix. Then A is singular if and only if $\det(A) = 0$.

Suppose that A is a square matrix of size n . The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
5. The columns of A are a linearly independent set.
6. A is invertible.
7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
8. The columns of A are a basis for \mathbb{C}^n .
9. The rank of A is n , $r(A) = n$.
10. The nullity of A is zero, $n(A) = 0$.
11. The determinant of A is nonzero, $\det(A) \neq 0$.

Suppose that A and B are square matrices of the same size. Then $\det(AB) = \det(A)\det(B)$.

Suppose that A is a square matrix of size n , $\mathbf{x} \neq \mathbf{0}$ is a vector in \mathbb{C}^n , and λ is a scalar in \mathbb{C} . Then we say \mathbf{x} is an **eigenvector** of A with **eigenvalue** λ if

$$A\mathbf{x} = \lambda\mathbf{x}$$

Suppose A is a square matrix. Then A has at least one eigenvalue.

Suppose that A is a square matrix of size n . Then the **characteristic polynomial** of A is the polynomial $p_A(x)$ defined by

$$p_A(x) = \det(A - xI_n)$$

Suppose A is a square matrix. Then λ is an eigenvalue of A if and only if $p_A(\lambda) = 0$.

Suppose that A is a square matrix and λ is an eigenvalue of A . Then the **eigenspace** of A for λ , $\mathcal{E}_A(\lambda)$, is the set of all the eigenvectors of A for λ , together with the inclusion of the zero vector.

Suppose A is a square matrix of size n and λ is an eigenvalue of A . Then the eigenspace $\mathcal{E}_A(\lambda)$ is a subspace of the vector space \mathbb{C}^n .

Suppose A is a square matrix of size n and λ is an eigenvalue of A . Then

$$\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I_n)$$

Suppose that A is a square matrix and λ is an eigenvalue of A . Then the **algebraic multiplicity** of λ , $\alpha_A(\lambda)$, is the highest power of $(x - \lambda)$ that divides the characteristic polynomial, $p_A(x)$.

Suppose that A is a square matrix and λ is an eigenvalue of A . Then the **geometric multiplicity** of λ , $\gamma_A(\lambda)$, is the dimension of the eigenspace $\mathcal{E}_A(\lambda)$.

Theorem EDELI Eigenvectors with Distinct Eigenvalues are Linearly Independent
245

Suppose that A is an $n \times n$ square matrix and $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_p\}$ is a set of eigenvectors with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_p$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Then S is a linearly independent set.

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Theorem SMZE Singular Matrices have Zero Eigenvalues

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Suppose A is a square matrix. Then A is singular if and only if $\lambda = 0$ is an eigenvalue of A .

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Suppose that A is a square matrix of size n . The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
5. The columns of A are a linearly independent set.
6. A is invertible.
7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
8. The columns of A are a basis for \mathbb{C}^n .
9. The rank of A is n , $r(A) = n$.
10. The nullity of A is zero, $n(A) = 0$.
11. The determinant of A is nonzero, $\det(A) \neq 0$.
12. $\lambda = 0$ is not an eigenvalue of A .

Suppose A is a square matrix and λ is an eigenvalue of A . Then $\alpha\lambda$ is an eigenvalue of αA .

Suppose A is a square matrix, λ is an eigenvalue of A , and $s \geq 0$ is an integer. Then λ^s is an eigenvalue of A^s .

Suppose A is a square matrix and λ is an eigenvalue of A . Let $q(x)$ be a polynomial in the variable x . Then $q(\lambda)$ is an eigenvalue of the matrix $q(A)$.

Suppose A is a square nonsingular matrix and λ is an eigenvalue of A . Then $\frac{1}{\lambda}$ is an eigenvalue of the matrix A^{-1} .

Suppose A is a square matrix and λ is an eigenvalue of A . Then λ is an eigenvalue of the matrix A^t .

Suppose A is a square matrix with real entries and \mathbf{x} is an eigenvector of A for the eigenvalue λ . Then $\bar{\mathbf{x}}$ is an eigenvector of A for the eigenvalue $\bar{\lambda}$.

Suppose that A is a square matrix of size n . Then the characteristic polynomial of A , $p_A(x)$, has degree n .

Suppose that A is a square matrix of size n with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k$. Then

$$\sum_{i=1}^k \alpha_A(\lambda_i) = n$$

Suppose that A is a square matrix of size n and λ is an eigenvalue. Then

$$1 \leq \gamma_A(\lambda) \leq \alpha_A(\lambda) \leq n$$

Suppose that A is a square matrix of size n . Then A cannot have more than n distinct eigenvalues.

Suppose that A is a Hermitian matrix and λ is an eigenvalue of A . Then $\lambda \in \mathbb{R}$.

Suppose that A is a Hermitian matrix and \mathbf{x} and \mathbf{y} are two eigenvectors of A for different eigenvalues. Then \mathbf{x} and \mathbf{y} are orthogonal vectors.

Suppose A and B are two square matrices of size n . Then A and B are **similar** if there exists a nonsingular matrix of size n , S , such that $A = S^{-1}BS$.

Suppose A , B and C are square matrices of size n . Then

1. A is similar to A . (Reflexive)
2. If A is similar to B , then B is similar to A . (Symmetric)
3. If A is similar to B and B is similar to C , then A is similar to C . (Transitive)

Suppose A and B are similar matrices. Then the characteristic polynomials of A and B are equal, that is, $p_A(x) = p_B(x)$.

Suppose that A is a square matrix. Then A is a **diagonal matrix** if $[A]_{ij} = 0$ whenever $i \neq j$.

Suppose A is a square matrix. Then A is **diagonalizable** if A is similar to a diagonal matrix.

Suppose A is a square matrix of size n . Then A is diagonalizable if and only if there exists a linearly independent set S that contains n eigenvectors of A .

Suppose A is a square matrix. Then A is diagonalizable if and only if $\gamma_A(\lambda) = \alpha_A(\lambda)$ for every eigenvalue λ of A .

Suppose A is a square matrix of size n with n distinct eigenvalues. Then A is diagonalizable.

A **linear transformation**, $T: U \mapsto V$, is a function that carries elements of the vector space U (called the **domain**) to the vector space V (called the **codomain**), and which has two additional properties

1. $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$ for all $\mathbf{u}_1, \mathbf{u}_2 \in U$
2. $T(\alpha\mathbf{u}) = \alpha T(\mathbf{u})$ for all $\mathbf{u} \in U$ and all $\alpha \in \mathbb{C}$

Suppose $T: U \mapsto V$ is a linear transformation. Then $T(\mathbf{0}) = \mathbf{0}$.

Suppose that A is an $m \times n$ matrix. Define a function $T: \mathbb{C}^n \mapsto \mathbb{C}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. Then T is a linear transformation.

Suppose that $T: \mathbb{C}^n \mapsto \mathbb{C}^m$ is a linear transformation. Then there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

Suppose that $T: U \mapsto V$ is a linear transformation, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t$ are vectors from U and $a_1, a_2, a_3, \dots, a_t$ are scalars from \mathbb{C} . Then

$$T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_t\mathbf{u}_t) = a_1T(\mathbf{u}_1) + a_2T(\mathbf{u}_2) + a_3T(\mathbf{u}_3) + \cdots + a_tT(\mathbf{u}_t)$$

Suppose that $T: U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a basis for U . Then for any vector $\mathbf{w} \in U$ there are unique scalars, $a_1, a_2, a_3, \dots, a_n$, from \mathbb{C} such that

$$\begin{aligned}\mathbf{w} &= a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_n\mathbf{u}_n \\ T(\mathbf{w}) &= a_1T(\mathbf{u}_1) + a_2T(\mathbf{u}_2) + a_3T(\mathbf{u}_3) + \cdots + a_nT(\mathbf{u}_n)\end{aligned}$$

Suppose that $T: U \mapsto V$ is a linear transformation. For each \mathbf{v} , define the **pre-image** of \mathbf{v} to be the subset of U given by

$$T^{-1}(\mathbf{v}) = \{\mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{v}\}$$

Suppose that $T: U \mapsto V$ and $S: U \mapsto V$ are two linear transformations with the same domain and codomain. Then their **sum** is the function $T + S: U \mapsto V$ whose outputs are defined by

$$(T + S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$$

Suppose that $T: U \mapsto V$ and $S: U \mapsto V$ are two linear transformations with the same domain and codomain. Then $T + S: U \mapsto V$ is a linear transformation.

Suppose that $T: U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then the **scalar multiple** is the function $\alpha T: U \mapsto V$ whose outputs are defined by

$$(\alpha T)(\mathbf{u}) = \alpha T(\mathbf{u})$$

Theorem MLTLT Multiple of a Linear Transformation is a Linear Transformation
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Suppose that $T: U \mapsto V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then $(\alpha T): U \mapsto V$ is a linear transformation.

Suppose that U and V are vector spaces. Then the set of all linear transformations from U to V , $\mathcal{LT}(U, V)$ is a vector space when the operations are those given in Definition LTA and Definition LTSM.

Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are linear transformations. Then the **composition** of S and T is the function $(S \circ T): U \mapsto W$ whose outputs are defined by

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$

Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are linear transformations. Then $(S \circ T): U \mapsto W$ is a linear transformation.

Suppose $T: U \mapsto V$ is a linear transformation. Then T is **injective** if whenever $T(\mathbf{x}) = T(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$.

Suppose $T: U \mapsto V$ is a linear transformation. Then the **kernel** of T is the set

$$\mathcal{K}(T) = \{\mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{0}\}$$

Suppose that $T: U \mapsto V$ is a linear transformation. Then the kernel of T , $\mathcal{K}(T)$, is a subspace of U .

Suppose $T: U \mapsto V$ is a linear transformation and $\mathbf{v} \in V$. If the preimage $T^{-1}(\mathbf{v})$ is non-empty, and $\mathbf{u} \in T^{-1}(\mathbf{v})$ then

$$T^{-1}(\mathbf{v}) = \{\mathbf{u} + \mathbf{z} \mid \mathbf{z} \in \mathcal{K}(T)\} = \mathbf{u} + \mathcal{K}(T)$$

Suppose that $T: U \mapsto V$ is a linear transformation. Then T is injective if and only if the kernel of T is trivial, $\mathcal{K}(T) = \{\mathbf{0}\}$.

Theorem ILTLI Injective Linear Transformations and Linear Independence 287

Suppose that $T: U \mapsto V$ is an injective linear transformation and $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$ is a linearly independent subset of U . Then $R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_t)\}$ is a linearly independent subset of V .

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Theorem ILTB Injective Linear Transformations and Bases 288

Suppose that $T: U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$ is a basis of U . Then T is injective if and only if $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$ is a linearly independent subset of V .

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Suppose that $T: U \mapsto V$ is an injective linear transformation. Then $\dim(U) \leq \dim(V)$.

Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are injective linear transformations. Then $(S \circ T): U \mapsto W$ is an injective linear transformation.

Suppose $T: U \mapsto V$ is a linear transformation. Then T is **surjective** if for every $\mathbf{v} \in V$ there exists a $\mathbf{u} \in U$ so that $T(\mathbf{u}) = \mathbf{v}$.

Suppose $T: U \mapsto V$ is a linear transformation. Then the **range** of T is the set

$$\mathcal{R}(T) = \{T(\mathbf{u}) \mid \mathbf{u} \in U\}$$

Suppose that $T: U \mapsto V$ is a linear transformation. Then the range of T , $\mathcal{R}(T)$, is a subspace of V .

Suppose that $T: U \mapsto V$ is a linear transformation. Then T is surjective if and only if the range of T equals the codomain, $\mathcal{R}(T) = V$.

Suppose that $T: U \mapsto V$ is a linear transformation and $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$ spans U . Then

$$R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_t)\}$$

spans $\mathcal{R}(T)$.

Suppose that $T: U \mapsto V$ is a linear transformation. Then

$$\mathbf{v} \in \mathcal{R}(T) \text{ if and only if } T^{-1}(\mathbf{v}) \neq \emptyset$$

Suppose that $T: U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$ is a basis of U . Then T is surjective if and only if $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$ is a spanning set for V .

Suppose that $T: U \mapsto V$ is a surjective linear transformation. Then $\dim(U) \geq \dim(V)$.

Theorem CSLTS **Composition of Surjective Linear Transformations is Surjective**
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Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are surjective linear transformations. Then $(S \circ T): U \mapsto W$ is a surjective linear transformation.

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Definition IDLT **Identity Linear Transformation**

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The **identity linear transformation** on the vector space W is defined as

$$I_W: W \mapsto W, \quad I_W(\mathbf{w}) = \mathbf{w}$$

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Suppose that $T: U \mapsto V$ is a linear transformation. If there is a function $S: V \mapsto U$ such that

$$S \circ T = I_U \qquad T \circ S = I_V$$

then T is **invertible**. In this case, we call S the **inverse** of T and write $S = T^{-1}$.

Theorem ILTLT Inverse of a Linear Transformation is a Linear Transformation
302

Suppose that $T: U \mapsto V$ is an invertible linear transformation. Then the function $T^{-1}: V \mapsto U$ is a linear transformation.

Suppose that $T: U \mapsto V$ is an invertible linear transformation. Then T^{-1} is an invertible linear transformation and $(T^{-1})^{-1} = T$.

Suppose $T: U \mapsto V$ is a linear transformation. Then T is invertible if and only if T is injective and surjective.

Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are invertible linear transformations. Then the composition, $(S \circ T): U \mapsto W$ is an invertible linear transformation.

Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are invertible linear transformations. Then $S \circ T$ is invertible and $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$.

Two vector spaces U and V are **isomorphic** if there exists an invertible linear transformation T with domain U and codomain V , $T: U \rightarrow V$. In this case, we write $U \cong V$, and the linear transformation T is known as an **isomorphism** between U and V .

Suppose U and V are isomorphic vector spaces. Then $\dim(U) = \dim(V)$.

Suppose that $T: U \mapsto V$ is a linear transformation. Then the **rank** of T , $r(T)$, is the dimension of the range of T ,

$$r(T) = \dim(\mathcal{R}(T))$$

Suppose that $T: U \mapsto V$ is a linear transformation. Then the **nullity** of T , $n(T)$, is the dimension of the kernel of T ,

$$n(T) = \dim(\mathcal{K}(T))$$

Suppose that $T: U \mapsto V$ is a linear transformation. Then the rank of T is the dimension of V , $r(T) = \dim(V)$, if and only if T is surjective.

Suppose that $T: U \mapsto V$ is an injective linear transformation. Then the nullity of T is zero, $n(T) = 0$, if and only if T is injective.

Suppose that $T: U \mapsto V$ is a linear transformation. Then

$$r(T) + n(T) = \dim(U)$$

Definition VR Vector Representation

Suppose that V is a vector space with a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$. Define a function $\rho_B: V \mapsto \mathbb{C}^n$ as follows. For $\mathbf{w} \in V$, find scalars $a_1, a_2, a_3, \dots, a_n$ so that

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_n\mathbf{v}_n$$

then

$$[\rho_B(\mathbf{w})]_i = a_i \quad 1 \leq i \leq n$$

The function ρ_B (Definition VR) is a linear transformation.

The function ρ_B (Definition VR) is an injective linear transformation.

The function ρ_B (Definition VR) is a surjective linear transformation.

The function ρ_B (Definition VR) is an invertible linear transformation.

Suppose that V is a vector space with dimension n . Then V is isomorphic to \mathbb{C}^n .

Suppose U and V are both finite-dimensional vector spaces. Then U and V are isomorphic if and only if $\dim(U) = \dim(V)$.

Suppose that U is a vector space with a basis B of size n . Then $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_k\}$ is a linearly independent subset of U if and only if $R = \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \dots, \rho_B(\mathbf{u}_k)\}$ is a linearly independent subset of \mathbb{C}^n .

Suppose that U is a vector space with a basis B of size n . Then $\mathbf{u} \in \langle\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_k\}\rangle$ if and only if $\rho_B(\mathbf{u}) \in \langle\{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \dots, \rho_B(\mathbf{u}_k)\}\rangle$.

Suppose that $T: U \mapsto V$ is a linear transformation, $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a basis for U of size n , and C is a basis for V of size m . Then the **matrix representation** of T relative to B and C is the $m \times n$ matrix,

$$M_{B,C}^T = [\rho_C(T(\mathbf{u}_1)) | \rho_C(T(\mathbf{u}_2)) | \rho_C(T(\mathbf{u}_3)) | \dots | \rho_C(T(\mathbf{u}_n))]$$

Suppose that $T: U \mapsto V$ is a linear transformation, B is a basis for U , C is a basis for V and $M_{B,C}^T$ is the matrix representation of T relative to B and C . Then, for any $\mathbf{u} \in U$,

$$\rho_C(T(\mathbf{u})) = M_{B,C}^T(\rho_B(\mathbf{u}))$$

or equivalently

$$T(\mathbf{u}) = \rho_C^{-1}(M_{B,C}^T(\rho_B(\mathbf{u})))$$

Theorem MRSLT Matrix Representation of a Sum of Linear Transformations 325

Suppose that $T: U \mapsto V$ and $S: U \mapsto V$ are linear transformations, B is a basis of U and C is a basis of V . Then

$$M_{B,C}^{T+S} = M_{B,C}^T + M_{B,C}^S$$

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Theorem MRMLT Matrix Representation of a Multiple of a Linear Transformation 326

Suppose that $T: U \mapsto V$ is a linear transformation, $\alpha \in \mathbb{C}$, B is a basis of U and C is a basis of V . Then

$$M_{B,C}^{\alpha T} = \alpha M_{B,C}^T$$

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Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are linear transformations, B is a basis of U , C is a basis of V , and D is a basis of W . Then

$$M_{B,D}^{S \circ T} = M_{C,D}^S M_{B,C}^T$$

Suppose that $T: U \mapsto V$ is a linear transformation, B is a basis for U of size n , and C is a basis for V . Then the kernel of T is isomorphic to the null space of $M_{B,C}^T$,

$$\mathcal{K}(T) \cong \mathcal{N}(M_{B,C}^T)$$

Suppose that $T: U \mapsto V$ is a linear transformation, B is a basis for U of size n , and C is a basis for V of size m . Then the range of T is isomorphic to the column space of $M_{B,C}^T$,

$$\mathcal{R}(T) \cong \mathcal{C}(M_{B,C}^T)$$

Suppose that $T: U \mapsto V$ is an invertible linear transformation, B is a basis for U and C is a basis for V . Then the matrix representation of T relative to B and C , $M_{B,C}^T$ is an invertible matrix, and

$$M_{C,B}^{T^{-1}} = (M_{B,C}^T)^{-1}$$

Suppose that A is a square matrix of size n and $T: \mathbb{C}^n \mapsto \mathbb{C}^n$ is the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. Then A is invertible matrix if and only if T is an invertible linear transformation.

Suppose that A is a square matrix of size n . The following are equivalent.

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
5. The columns of A are a linearly independent set.
6. A is invertible.
7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
8. The columns of A are a basis for \mathbb{C}^n .
9. The rank of A is n , $r(A) = n$.
10. The nullity of A is zero, $n(A) = 0$.
11. The determinant of A is nonzero, $\det(A) \neq 0$.
12. $\lambda = 0$ is not an eigenvalue of A .
13. The linear transformation $T: \mathbb{C}^n \mapsto \mathbb{C}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is invertible.

Suppose that $T: V \mapsto V$ is a linear transformation. Then a nonzero vector $\mathbf{v} \in V$ is an **eigenvector** of T for the **eigenvalue** λ if $T(\mathbf{v}) = \lambda\mathbf{v}$.

Suppose that V is a vector space, and $I_V: V \mapsto V$ is the identity linear transformation on V . Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$ and C be two bases of V . Then the **change-of-basis matrix** from B to C is the matrix representation of I_V relative to B and C ,

$$\begin{aligned} C_{B,C} &= M_{B,C}^{I_V} \\ &= [\rho_C(I_V(\mathbf{v}_1)) \mid \rho_C(I_V(\mathbf{v}_2)) \mid \rho_C(I_V(\mathbf{v}_3)) \mid \dots \mid \rho_C(I_V(\mathbf{v}_n))] \\ &= [\rho_C(\mathbf{v}_1) \mid \rho_C(\mathbf{v}_2) \mid \rho_C(\mathbf{v}_3) \mid \dots \mid \rho_C(\mathbf{v}_n)] \end{aligned}$$

Suppose that \mathbf{v} is a vector in the vector space V and B and C are bases of V . Then

$$\rho_C(\mathbf{v}) = C_{B,C}\rho_B(\mathbf{v})$$

Suppose that V is a vector space, and B and C are bases of V . Then the change-of-basis matrix $C_{B,C}$ is nonsingular and

$$C_{B,C}^{-1} = C_{C,B}$$

Suppose that B and C are orthonormal bases for \mathbb{C}^n . Then the change-of-basis matrix, $C_{B,C}$, is a unitary matrix.

Suppose that $T: U \mapsto V$ is a linear transformation, B and C are bases for U , and D and E are bases for V . Then

$$M_{B,D}^T = C_{E,D} M_{C,E}^T C_{B,C}$$

Suppose that $T: V \mapsto V$ is a linear transformation and B and C are bases of V . Then

$$M_{B,B}^T = C_{B,C}^{-1} M_{C,C}^T C_{B,C}$$

Suppose that $T: V \mapsto V$ is a linear transformation and B is a basis of V . Then $\mathbf{v} \in V$ is an eigenvector of T for the eigenvalue λ if and only if $\rho_B(\mathbf{v})$ is an eigenvector of $M_{B,B}^T$ for the eigenvalue λ .

Suppose that $T: V \mapsto V$ is a linear transformation such that there is an integer $p > 0$ such that $T^p(\mathbf{v}) = \mathbf{0}$ for every $\mathbf{v} \in V$. The smallest p for which this condition is met is called the **index** of T .

Given the scalar $\lambda \in \mathbb{C}$, the Jordan block $J_n(\lambda)$ is the $n \times n$ matrix defined by

$$[J_n(\lambda)]_{ij} = \begin{cases} \lambda & i = j \\ 1 & j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

The Jordan block $J_n(0)$ is nilpotent of index n .

Suppose that $T: V \mapsto V$ is a nilpotent linear transformation and λ is an eigenvalue of T . Then $\lambda = 0$.

Suppose the linear transformation $T: V \mapsto V$ is nilpotent. Then T is diagonalizable if and only if T is the zero linear transformation.

Suppose $T: V \mapsto V$ is a linear transformation, where $\dim(V) = n$. Then there is an integer m , $0 \leq m \leq n$, such that

$$\{\mathbf{0}\} = \mathcal{K}(T^0) \subsetneq \mathcal{K}(T^1) \subsetneq \mathcal{K}(T^2) \subsetneq \cdots \subsetneq \mathcal{K}(T^m) = \mathcal{K}(T^{m+1}) = \mathcal{K}(T^{m+2}) = \cdots$$

Theorem KPFLT Kernels of Powers of Nilpotent Linear Transformations 347

Suppose $T: V \mapsto V$ is a nilpotent linear transformation with index p and $\dim(V) = n$. Then $0 \leq p \leq n$ and

$$\{0\} = \mathcal{K}(T^0) \subsetneq \mathcal{K}(T^1) \subsetneq \mathcal{K}(T^2) \subsetneq \cdots \subsetneq \mathcal{K}(T^p) = \mathcal{K}(T^{p+1}) = \cdots = V$$

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Theorem CFNLT Canonical Form for Nilpotent Linear Transformations 348

Suppose that $T: V \mapsto V$ is a nilpotent linear transformation of index p . Then there is a basis for V so that the matrix representation, $M_{B,B}^T$, is block diagonal with each block being a Jordan block, $J_n(0)$. The size of the largest block is the index p , and the total number of blocks is the nullity of T , $n(T)$.

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Suppose that $T: V \mapsto V$ is a linear transformation and W is a subspace of V . Suppose further that $T(\mathbf{w}) \in W$ for every $\mathbf{w} \in W$. Then W is an **invariant subspace** of V relative to T .

Suppose that $T: V \mapsto V$ is a linear transformation with eigenvalue λ and associated eigenspace $\mathcal{E}_T(\lambda)$. Let W be any subspace of $\mathcal{E}_T(\lambda)$. Then W is an invariant subspace of V relative to T .

Suppose that $T: V \mapsto V$ is a linear transformation. Then $\mathcal{K}(T^k)$ is an invariant subspace of V .

Suppose that $T: V \mapsto V$ is a linear transformation. Suppose further that for $\mathbf{x} \neq \mathbf{0}$, $(T - \lambda I_V)^k(\mathbf{x}) = \mathbf{0}$ for some $k > 0$. Then \mathbf{x} is a **generalized eigenvector** of T with eigenvalue λ .

Suppose that $T: V \mapsto V$ is a linear transformation. Define the **generalized eigenspace** of T for λ as

$$\mathcal{G}_T(\lambda) = \left\{ \mathbf{x} \mid (T - \lambda I_V)^k(\mathbf{x}) = \mathbf{0} \text{ for some } k \geq 0 \right\}$$

Suppose that $T: V \mapsto V$ is a linear transformation. Then the generalized eigenspace $\mathcal{G}_T(\lambda)$ is an invariant subspace of V relative to T .

Suppose that $T: V \mapsto V$ is a linear transformation, $\dim(V) = n$, and λ is an eigenvalue of T . Then $\mathcal{G}_T(\lambda) = \mathcal{K}((T - \lambda I_V)^n)$.

Suppose that $T: V \mapsto V$ is a linear transformation, and U is an invariant subspace of V relative to T . Define the **restriction** of T to U by

$$T|_U: U \mapsto U \qquad T|_U(\mathbf{u}) = T(\mathbf{u})$$

Suppose $T: V \mapsto V$ is a linear transformation with eigenvalue λ . Then the linear transformation $T|_{\mathcal{G}_T(\lambda)} - \lambda I_{\mathcal{G}_T(\lambda)}$ is nilpotent.

Suppose $T: V \mapsto V$ is a linear transformation with eigenvalue λ . Then the **index** of λ , $\iota_T(\lambda)$, is the index of the nilpotent linear transformation $T|_{\mathcal{G}_T(\lambda)} - \lambda I_{\mathcal{G}_T(\lambda)}$.

Theorem MRRGE Matrix Representation of a Restriction to a Generalized Eigenspace **359**

Suppose that $T: V \mapsto V$ is a linear transformation with eigenvalue λ . Then there is a basis of the the generalized eigenspace $\mathcal{G}_T(\lambda)$ such that the restriction $T|_{\mathcal{G}_T(\lambda)}: \mathcal{G}_T(\lambda) \mapsto \mathcal{G}_T(\lambda)$ has a matrix representation that is block diagonal where each block is a Jordan block of the form $J_n(\lambda)$.

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Theorem GESD Generalized Eigenspace Decomposition **360**

Suppose that $T(V) V$ is a linear transformation with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_m$. Then

$$V = \mathcal{G}_T(\lambda_1) \oplus \mathcal{G}_T(\lambda_2) \oplus \mathcal{G}_T(\lambda_3) \oplus \cdots \oplus \mathcal{G}_T(\lambda_m)$$

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Suppose $T: V \mapsto V$ is a linear transformation with eigenvalue λ . Then the dimension of the generalized eigenspace for λ is the algebraic multiplicity of λ , $\dim(\mathcal{G}_T(\lambda)) = \alpha_T(\lambda)$.

A square matrix is in **Jordan canonical form** if it meets the following requirements:

1. The matrix is block diagonal.
2. Each block is a Jordan block.
3. If $\rho < \lambda$ then the block $J_k(\rho)$ occupies rows with indices greater than the indices of the rows occupied by $J_\ell(\lambda)$.
4. If $\rho = \lambda$ and $\ell < k$, then the block $J_\ell(\lambda)$ occupies rows with indices greater than the indices of the rows occupied by $J_k(\lambda)$.

Suppose $T: V \mapsto V$ is a linear transformation. Then there is a basis B for V such that the matrix representation of T with the following properties:

1. The matrix representation is in Jordan canonical form.
2. If $J_k(\lambda)$ is one of the Jordan blocks, then λ is an eigenvalue of T .
3. For a fixed value of λ , the largest block of the form $J_k(\lambda)$ has size equal to the index of λ , $\iota_T(\lambda)$.
4. For a fixed value of λ , the number of blocks of the form $J_k(\lambda)$ is the geometric multiplicity of λ , $\gamma_T(\lambda)$.
5. For a fixed value of λ , the number of rows occupied by blocks of the form $J_k(\lambda)$ is the algebraic multiplicity of λ , $\alpha_T(\lambda)$.

Suppose A is a square matrix with characteristic polynomial $p_A(x)$. Then $p_A(A) = \mathcal{O}$.

