Exercise and Solution Manual
for
A First Course in Linear Algebra

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Section WILA
What is Linear Algebra?

C10 (Robert Beezer) In Example TMP the first table lists the cost (per kilogram) to manufacture each of the three varieties of trail mix (bulk, standard, fancy). For example, it costs \$3.69 to make one kilogram of the bulk variety. Re-compute each of these three costs and notice that the computations are linear in character.

M70 (Robert Beezer) In Example TMP two different prices were considered for marketing standard mix with the revised recipes (one-third peanuts in each recipe). Selling standard mix at \$5.50 resulted in selling the minimum amount of the fancy mix and no bulk mix. At \$5.25 it was best for profits to sell the maximum amount of fancy mix and then sell no standard mix. Determine a selling price for standard mix that allows for maximum profits while still selling some of each type of mix.

Solution (Robert Beezer) If the price of standard mix is set at \$5.292, then the profit function has a zero coefficient on the variable quantity f. So, we can set f to be any integer quantity in $\{825, 826, \ldots, 960\}$. All but the extreme values ($f = 825$, $f = 960$) will result in production levels where some of every mix is manufactured. No matter what value of f is chosen, the resulting profit will be the same, at \$2,664.60.

Section SSLE
Solving Systems of Linear Equations

C10 (Robert Beezer) Find a solution to the system in Example IS where $x_3 = 6$ and $x_4 = 2$. Find two other solutions to the system. Find a solution where $x_1 = -17$ and $x_2 = 14$. How many possible answers are there to each of these questions?

C20 (Robert Beezer) Each archetype (Archetypes) that is a system of equations begins by listing some specific solutions. Verify the specific solutions listed in the following archetypes by evaluating the system of equations with the solutions listed.

Archetype A, Archetype B, Archetype C, Archetype D, Archetype E, Archetype F, Archetype G, Archetype H, Archetype I, Archetype J

C30 (Chris Black) Find all solutions to the linear system:

$$\begin{aligned} x + y &= 5 \\ 2x - y &= 3 \end{aligned}$$

Solution (Chris Black) Solving each equation for y, we have the equivalent system

$$\begin{aligned} y &= 5 - x \\ y &= 2x - 3. \end{aligned}$$

Setting these expressions for y equal, we have the equation $5 - x = 2x - 3$, which quickly leads to $x = \frac{8}{3}$. Substituting for x in the first equation, we have $y = 5 - x = 5 - \frac{8}{3} = \frac{7}{3}$. Thus, the solution is $x = \frac{8}{3}$, $y = \frac{7}{3}$.

C31 (Chris Black) Find all solutions to the linear system:

$$\begin{aligned} 3x + 2y &= 1 \\ x - y &= 2 \\ 4x + 2y &= 2 \end{aligned}$$

C32 (Chris Black) Find all solutions to the linear system:

$$\begin{aligned} x + 2y &= 8 \\ x - y &= 2 \\ x + y &= 4 \end{aligned}$$

C33 (Chris Black) Find all solutions to the linear system:

$$\begin{aligned} x + y - z &= -1 \\ x - y - z &= -1 \\ z &= 2 \end{aligned}$$

C34 (Chris Black) Find all solutions to the linear system:

$$\begin{aligned} x + y - z &= -5 \\ x - y - z &= -3 \\ x + y - z &= 0 \end{aligned}$$

C50 (Robert Beezer) A three-digit number has two properties. The tens-digit and the ones-digit add up to 5. If the number is written with the digits in the reverse order, and then subtracted from the original number, the result is 792. Use a system of equations to find all of the three-digit numbers with these properties.

Solution (Robert Beezer) Let a be the hundreds digit, b the tens digit, and c the ones digit. Then the first condition says that $b + c = 5$. The original number is $100a + 10b + c$, while the reversed number is $100c + 10b + a$. So the second condition is

$$792 = (100a + 10b + c) - (100c + 10b + a) = 99a - 99c$$

So we arrive at the system of equations

$$\begin{aligned} b + c &= 5 \\ 99a - 99c &= 792 \end{aligned}$$

Using equation operations, we arrive at the equivalent system

$$\begin{aligned} a - c &= 8 \\ b + c &= 5 \end{aligned}$$

We can vary c and obtain infinitely many solutions. However, c must be a digit, restricting us to ten values $(0 - 9)$. Furthermore, if $c > 1$, then the first equation forces $a > 9$, an impossibility. Setting $c = 0$, yields 850 as a solution, and setting $c = 1$ yields 941 as another solution.

C51 (Robert Beezer) Find all of the six-digit numbers in which the first digit is one less than the second, the third digit is half the second, the fourth digit is three times the third and the last two digits form a number that equals the sum of the fourth and fifth. The sum of all the digits is 24. (From *The MENSA Puzzle Calendar* for January 9, 2006.)

Solution (Robert Beezer) Let $abcdef$ denote any such six-digit number and convert each requirement in the problem statement into an equation.

$$a = b - 1$$

$$
\begin{aligned}
c &= \frac{1}{2}b \\
d &= 3c \\
10e + f &= d + e \\
24 &= a + b + c + d + e + f
\end{aligned}
$$

In a more standard form this becomes

$$
\begin{aligned}
a - b &= -1 \\
-b + 2c &= 0 \\
-3c + d &= 0 \\
-d + 9e + f &= 0 \\
a + b + c + d + e + f &= 24
\end{aligned}
$$

Using equation operations (or the techniques of the upcoming Section RREF), this system can be converted to the equivalent system

$$
\begin{aligned}
a + \frac{16}{75}f &= 5 \\
b + \frac{16}{75}f &= 6 \\
c + \frac{8}{75}f &= 3 \\
d + \frac{8}{25}f &= 9 \\
e + \frac{11}{75}f &= 1
\end{aligned}
$$

Clearly, choosing $f = 0$ will yield the solution $abcde = 563910$. Furthermore, to have the variables result in single-digit numbers, none of the other choices for f (1, 2, ..., 9) will yield a solution.

C52 (Robert Beezer) Driving along, Terry notices that the last four digits on his car's odometer are palindromic. A mile later, the last five digits are palindromic. After driving another mile, the middle four digits are palindromic. One more mile, and all six are palindromic. What was the odometer reading when Terry first looked at it? Form a linear system of equations that expresses the requirements of this puzzle. (*Car Talk* Puzzler, National Public Radio, Week of January 21, 2008) (A car odometer displays six digits and a sequence is a **palindrome** if it reads the same left-to-right as right-to-left.)

Solution (Robert Beezer) 198888 is one solution, and David Braithwaite found 199999 as another.

M10 (Robert Beezer) Each sentence below has at least two meanings. Identify the source of the double meaning, and rewrite the sentence (at least twice) to clearly convey each meaning.

1. They are baking potatoes.
2. He bought many ripe pears and apricots.
3. She likes his sculpture.
4. I decided on the bus.

Solution (Robert Beezer)

1. Does "baking" describe the potato or what is happening to the potato?
   Those are potatoes that are used for baking.
   The potatoes are being baked.
2. Are the apricots ripe, or just the pears? Parentheses could indicate just what the adjective "ripe" is meant to modify. Were there many apricots as well, or just many pears?
   He bought many pears and many ripe apricots.
   He bought apricots and many ripe pears.

3. Is "sculpture" a single physical object, or the sculptor's style expressed over many pieces and many years?
   She likes his sculpture of the girl.
   She likes his sculptural style.

4. Was a decision made while in the bus, or was the outcome of a decision to choose the bus. Would the sentence "I decided on the car," have a similar double meaning?
   I made my decision while on the bus.
   I decided to ride the bus.

M11 (Robert Beezer) Discuss the difference in meaning of each of the following three almost identical sentences, which all have the same grammatical structure. (These are due to Keith Devlin.)

1. She saw him in the park with a dog.

2. She saw him in the park with a fountain.

3. She saw him in the park with a telescope.

Solution (Robert Beezer) We know the dog belongs to the man, and the fountain belongs to the park. It is not clear if the telescope belongs to the man, the woman, or the park.

M12 (Robert Beezer) The following sentence, due to Noam Chomsky, has a correct grammatical structure, but is meaningless. Critique its faults. "Colorless green ideas sleep furiously." (Chomsky, Noam. *Syntactic Structures*, The Hague/Paris: Mouton, 1957. p. 15.)

Solution (Robert Beezer) In adjacent pairs the words are contradictory or inappropriate. Something cannot be both green and colorless, ideas do not have color, ideas do not sleep, and it is hard to sleep furiously.

M13 (Robert Beezer) Read the following sentence and form a mental picture of the situation.
The baby cried and the mother picked it up.
What *assumptions* did you make about the situation?

Solution (Robert Beezer) Did you assume that the baby and mother are human?
Did you assume that the baby is the child of the mother?
Did you assume that the mother picked up the baby as an attempt to stop the crying?

M14 (Robert Beezer) Discuss the difference in meaning of the following two almost identical sentences, which have nearly identical grammatical structure. (This antanaclasis is often attributed to the comedian Groucho Marx, but has earlier roots.)

1. Time flies like an arrow.

2. Fruit flies like a banana.

M30 (David Beezer) This problem appears in a middle-school mathematics textbook: Together Dan and Diane have \$20. Together Diane and Donna have \$15. How much do the three of them have in total? (*Transition Mathematics*, Second Edition, Scott Foresman Addison Wesley, 1998. Problem 5–1.19.)

Solution (Robert Beezer) If x, y and z represent the money held by Dan, Diane and Donna, then $y = 15-z$ and $x = 20 - y = 20 - (15 - z) = 5 + z$. We can let z take on any value from 0 to 15 without any of the three amounts being negative, since presumably middle-schoolers are too young to assume debt.

Then the total capital held by the three is $x + y + z = (5 + z) + (15 - z) + z = 20 + z$. So their combined holdings can range anywhere from \$20 (Donna is broke) to \$35 (Donna is flush).

We will have more to say about this situation in Section TSS, and specifically Theorem CMVEI.

M40 (Robert Beezer) Solutions to the system in Example IS are given as

$$(x_1,\, x_2,\, x_3,\, x_4) = (-1 - 2a + 3b,\, 4 + a - 2b,\, a,\, b)$$

Evaluate the three equations of the original system with these expressions in a and b and verify that each equation is true, no matter what values are chosen for a and b.

M70 (Robert Beezer) We have seen in this section that systems of linear equations have limited possibilities for solution sets, and we will shortly prove Theorem PSSLS that describes these possibilities exactly. This exercise will show that if we relax the requirement that our equations be linear, then the possibilities expand greatly. Consider a system of two equations in the two variables x and y, where the departure from linearity involves simply squaring the variables.

$$\begin{aligned} x^2 - y^2 &= 1 \\ x^2 + y^2 &= 4 \end{aligned}$$

After solving this system of *non-linear* equations, replace the second equation in turn by $x^2 + 2x + y^2 = 3$, $x^2 + y^2 = 1$, $x^2 - 4x + y^2 = -3$, $-x^2 + y^2 = 1$ and solve each resulting system of two equations in two variables. (This exercise includes suggestions from Don Kreher.)

Solution (Robert Beezer) The equation $x^2 - y^2 = 1$ has a solution set by itself that has the shape of a hyperbola when plotted. Four of the five different second equations have solution sets that are circles when plotted individually (the last is another hyperbola). Where the hyperbola and circles intersect are the solutions to the system of two equations. As the size and location of the circles vary, the number of intersections varies from four to one (in the order given). The last equation is a hyperbola that "opens" in the other direction. Sketching the relevant equations would be instructive, as was discussed in Example STNE.

The exact solution sets are (according to the choice of the second equation),

$$\begin{aligned}
x^2 + y^2 = 4 : &\quad \left\{\left(\sqrt{\frac{5}{2}}, \sqrt{\frac{3}{2}}\right),\, \left(-\sqrt{\frac{5}{2}}, \sqrt{\frac{3}{2}}\right),\, \left(\sqrt{\frac{5}{2}}, -\sqrt{\frac{3}{2}}\right),\, \left(-\sqrt{\frac{5}{2}}, -\sqrt{\frac{3}{2}}\right)\right\} \\
x^2 + 2x + y^2 = 3 : &\quad \left\{(1, 0),\, (-2, \sqrt{3}),\, (-2, -\sqrt{3})\right\} \\
x^2 + y^2 = 1 : &\quad \{(1, 0),\, (-1, 0)\} \\
x^2 - 4x + y^2 = -3 : &\quad \{(1, 0)\} \\
-x^2 + y^2 = 1 : &\quad \{\}
\end{aligned}$$

T10 (Robert Beezer) Proof Technique D asks you to formulate a definition of what it means for a whole number to be odd. What is your definition? (Do not say "the opposite of even.") Is 6 odd? Is 11 odd? Justify your answers by using your definition.

Solution (Robert Beezer) We can say that an integer is **odd** if when it is divided by 2 there is a remainder of 1. So 6 is not odd since $6 = 3 \times 2 + 0$, while 11 is odd since $11 = 5 \times 2 + 1$.

T20 (Robert Beezer) Explain why the second equation operation in Definition EO requires that the scalar be nonzero, while in the third equation operation this restriction on the scalar is not present.

Solution (Robert Beezer) Definition EO is engineered to make Theorem EOPSS true. If we were to allow a zero scalar to multiply an equation then that equation would be transformed to the equation $0 = 0$, which is true for any possible values of the variables. Any restrictions on the solution set imposed by the original equation would be lost.

However, in the third operation, it is allowed to choose a zero scalar, multiply an equation by this scalar and add the transformed equation to a second equation (leaving the first unchanged). The result? Nothing. The second equation is the same as it was before. So the theorem is true in this case, the two systems are equivalent. But in practice, this would be a silly thing to actually ever do! We still allow it though, in order to keep our theorem as general as possible.

Notice the location in the proof of Theorem EOPSS where the expression $\frac{1}{\alpha}$ appears — this explains the prohibition on $\alpha = 0$ in the second equation operation.

Section RREF
Reduced Row-Echelon Form

C05 (Robert Beezer) Each archetype below is a system of equations. Form the augmented matrix of the system of equations, convert the matrix to reduced row-echelon form by using equation operations and then describe the solution set of the original system of equations.

Archetype A, Archetype B, Archetype C, Archetype D, Archetype E, Archetype F, Archetype G, Archetype H, Archetype I, Archetype J

For problems C10–C19, find all solutions to the system of linear equations. Use your favorite computing device to row-reduce the augmented matrices for the systems, and write the solutions as a set, using correct set notation.

C10 (Robert Beezer)

$$\begin{aligned} 2x_1 - 3x_2 + x_3 + 7x_4 &= 14 \\ 2x_1 + 8x_2 - 4x_3 + 5x_4 &= -1 \\ x_1 + 3x_2 - 3x_3 &= 4 \\ -5x_1 + 2x_2 + 3x_3 + 4x_4 &= -19 \end{aligned}$$

Solution (Robert Beezer) The augmented matrix row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 0 & -3 \\ 0 & 0 & \boxed{1} & 0 & -4 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{bmatrix}$$

This augmented matrix represents the linear system $x_1 = 1$, $x_2 = -3$, $x_3 = -4$, $x_4 = 1$, which clearly has only one possible solution. We can write this solution set then as

$$S = \left\{ \begin{bmatrix} 1 \\ -3 \\ -4 \\ 1 \end{bmatrix} \right\}$$

C11 (Robert Beezer)

$$\begin{aligned} 3x_1 + 4x_2 - x_3 + 2x_4 &= 6 \\ x_1 - 2x_2 + 3x_3 + x_4 &= 2 \\ 10x_2 - 10x_3 - x_4 &= 1 \end{aligned}$$

Solution (Robert Beezer) The augmented matrix row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 1 & 4/5 & 0 \\ 0 & \boxed{1} & -1 & -1/10 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

Row 3 represents the equation $0 = 1$, which is patently false, so the original system has no solutions. We can express the solution set as the empty set, $\emptyset = \{\ \}$.

C12 (Robert Beezer)

$$\begin{aligned} 2x_1 + 4x_2 + 5x_3 + 7x_4 &= -26 \\ x_1 + 2x_2 + x_3 - x_4 &= -4 \\ -2x_1 - 4x_2 + x_3 + 11x_4 &= -10 \end{aligned}$$

Solution (Robert Beezer) The augmented matrix row-reduces to

$$\begin{bmatrix} \boxed{1} & 2 & 0 & -4 & 2 \\ 0 & 0 & \boxed{1} & 3 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In the spirit of Example SAA, we can express the infinitely many solutions of this system compactly with set notation. The key is to express certain variables in terms of others. More specifically, each pivot column number is the index of a variable that can be written in terms of the variables whose indices are non-pivot columns. Or saying the same thing: for each i in D, we can find an expression for x_i in terms of the variables without their index in D. Here $D = \{1, 3\}$, so

$$\begin{aligned} x_1 &= 2 - 2x_2 + 4x_4 \\ x_3 &= -6 \qquad - 3x_4 \end{aligned}$$

As a set, we write the solutions precisely as

$$\left\{ \left.\begin{bmatrix} 2 - 2x_2 + 4x_4 \\ x_2 \\ -6 - 3x_4 \\ x_4 \end{bmatrix} \right| x_2,\, x_4 \in \mathbb{C} \right\}$$

C13 (Robert Beezer)

$$\begin{aligned} x_1 + 2x_2 + 8x_3 - 7x_4 &= -2 \\ 3x_1 + 2x_2 + 12x_3 - 5x_4 &= 6 \\ -x_1 + x_2 + x_3 - 5x_4 &= -10 \end{aligned}$$

Solution (Robert Beezer) The augmented matrix of the system of equations is

$$\begin{bmatrix} 1 & 2 & 8 & -7 & -2 \\ 3 & 2 & 12 & -5 & 6 \\ -1 & 1 & 1 & -5 & -10 \end{bmatrix}$$

which row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 2 & 1 & 0 \\ 0 & \boxed{1} & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

Row 3 represents the equation $0 = 1$, which is patently false, so the original system has no solutions. We can express the solution set as the empty set, $\emptyset = \{\ \}$.

C14 (Robert Beezer)

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 2x_4 &= 4 \\ 3x_1 - 2x_2 + 11x_4 &= 13 \\ x_1 + x_2 + 5x_3 - 3x_4 &= 1 \end{aligned}$$

Solution (Robert Beezer) The augmented matrix of the system of equations is

$$\begin{bmatrix} 2 & 1 & 7 & -2 & 4 \\ 3 & -2 & 0 & 11 & 13 \\ 1 & 1 & 5 & -3 & 1 \end{bmatrix}$$

which row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 2 & 1 & 3 \\ 0 & \boxed{1} & 3 & -4 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In the spirit of Example SAA, we can express the infinitely many solutions of this system compactly with set notation. The key is to express certain variables in terms of others. More specifically, each pivot column number is the index of a variable that can be written in terms of the variables whose indices are non-pivot columns. Or saying the same thing: for each i in D, we can find an expression for x_i in terms of the variables without their index in D. Here $D = \{1, 2\}$, so rearranging the equations represented by the two nonzero rows to gain expressions for the variables x_1 and x_2 yields the solution set,

$$S = \left\{ \left.\begin{bmatrix} 3 - 2x_3 - x_4 \\ -2 - 3x_3 + 4x_4 \\ x_3 \\ x_4 \end{bmatrix} \right| x_3,\, x_4 \in \mathbb{C} \right\}$$

C15 (Robert Beezer)

$$\begin{aligned} 2x_1 + 3x_2 - x_3 - 9x_4 &= -16 \\ x_1 + 2x_2 + x_3 &= 0 \\ -x_1 + 2x_2 + 3x_3 + 4x_4 &= 8 \end{aligned}$$

Solution (Robert Beezer) The augmented matrix of the system of equations is

$$\begin{bmatrix} 2 & 3 & -1 & -9 & -16 \\ 1 & 2 & 1 & 0 & 0 \\ -1 & 2 & 3 & 4 & 8 \end{bmatrix}$$

which row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 2 & 3 \\ 0 & \boxed{1} & 0 & -3 & -5 \\ 0 & 0 & \boxed{1} & 4 & 7 \end{bmatrix}$$

In the spirit of Example SAA, we can express the infinitely many solutions of this system compactly with set notation. The key is to express certain variables in terms of others. More specifically, each pivot column number is the index of a variable that can be written in terms of the variables whose indices are non-pivot columns. Or saying the same thing: for each i in D, we can find an expression for x_i in terms of the variables without their index in D. Here $D = \{1, 2, 3\}$, so rearranging the equations represented by the three nonzero rows to gain expressions for the variables x_1, x_2 and x_3 yields the solution set,

$$S = \left\{ \begin{bmatrix} 3 - 2x_4 \\ -5 + 3x_4 \\ 7 - 4x_4 \\ x_4 \end{bmatrix} \middle| \, x_4 \in \mathbb{C} \right\}$$

C16 (Robert Beezer)

$$\begin{aligned} 2x_1 + 3x_2 + 19x_3 - 4x_4 &= 2 \\ x_1 + 2x_2 + 12x_3 - 3x_4 &= 1 \\ -x_1 + 2x_2 + 8x_3 - 5x_4 &= 1 \end{aligned}$$

Solution (Robert Beezer) The augmented matrix of the system of equations is

$$\begin{bmatrix} 2 & 3 & 19 & -4 & 2 \\ 1 & 2 & 12 & -3 & 1 \\ -1 & 2 & 8 & -5 & 1 \end{bmatrix}$$

which row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 2 & 1 & 0 \\ 0 & \boxed{1} & 5 & -2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

Row 3 represents the equation $0 = 1$, which is patently false, so the original system has no solutions. We can express the solution set as the empty set, $\emptyset = \{\ \}$.

C17 (Robert Beezer)

$$\begin{aligned} -x_1 + 5x_2 &= -8 \\ -2x_1 + 5x_2 + 5x_3 + 2x_4 &= 9 \\ -3x_1 - x_2 + 3x_3 + x_4 &= 3 \\ 7x_1 + 6x_2 + 5x_3 + x_4 &= 30 \end{aligned}$$

Solution (Robert Beezer) We row-reduce the augmented matrix of the system of equations,

$$\begin{bmatrix} -1 & 5 & 0 & 0 & -8 \\ -2 & 5 & 5 & 2 & 9 \\ -3 & -1 & 3 & 1 & 3 \\ 7 & 6 & 5 & 1 & 30 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & 0 & -1 \\ 0 & 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & 0 & \boxed{1} & 5 \end{bmatrix}$$

This augmented matrix represents the linear system $x_1 = 3$, $x_2 = -1$, $x_3 = 2$, $x_4 = 5$, which clearly has only one possible solution. We can write this solution set then as

$$S = \left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \\ 5 \end{bmatrix} \right\}$$

C18 (Robert Beezer)

$$\begin{aligned} x_1 + 2x_2 - 4x_3 - x_4 &= 32 \\ x_1 + 3x_2 - 7x_3 - x_5 &= 33 \\ x_1 + 2x_3 - 2x_4 + 3x_5 &= 22 \end{aligned}$$

Solution (Robert Beezer) We row-reduce the augmented matrix of the system of equations,

$$\begin{bmatrix} 1 & 2 & -4 & -1 & 0 & 32 \\ 1 & 3 & -7 & 0 & -1 & 33 \\ 1 & 0 & 2 & -2 & 3 & 22 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2 & 0 & 5 & 6 \\ 0 & \boxed{1} & -3 & 0 & -2 & 9 \\ 0 & 0 & 0 & \boxed{1} & 1 & -8 \end{bmatrix}$$

In the spirit of Example SAA, we can express the infinitely many solutions of this system compactly with set notation. The key is to express certain variables in terms of others. More specifically, each pivot column number is the index of a variable that can be written in terms of the variables whose indices are non-pivot columns. Or saying the same thing: for each i in D, we can find an expression for x_i in terms of the variables without their index in D. Here $D = \{1,\, 2,\, 4\}$, so

$$\begin{aligned} x_1 + 2x_3 + 5x_5 = 6 \quad &\to \quad x_1 = 6 - 2x_3 - 5x_5 \\ x_2 - 3x_3 - 2x_5 = 9 \quad &\to \quad x_2 = 9 + 3x_3 + 2x_5 \\ x_4 + x_5 = -8 \quad &\to \quad x_4 = -8 - x_5 \end{aligned}$$

As a set, we write the solutions precisely as

$$S = \left\{ \begin{bmatrix} 6 - 2x_3 - 5x_5 \\ 9 + 3x_3 + 2x_5 \\ x_3 \\ -8 - x_5 \\ x_5 \end{bmatrix} \middle|\, x_3,\, x_5 \in \mathbb{C} \right\}$$

C19 (Robert Beezer)

$$\begin{aligned} 2x_1 + x_2 &= 6 \\ -x_1 - x_2 &= -2 \\ 3x_1 + 4x_2 &= 4 \\ 3x_1 + 5x_2 &= 2 \end{aligned}$$

Solution (Robert Beezer) We form the augmented matrix of the system,

$$\begin{bmatrix} 2 & 1 & 6 \\ -1 & -1 & -2 \\ 3 & 4 & 4 \\ 3 & 5 & 2 \end{bmatrix}$$

which row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 4 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This augmented matrix represents the linear system $x_1 = 4$, $x_2 = -2$, $0 = 0$, $0 = 0$, which clearly has only one possible solution. We can write this solution set then as

$$S = \left\{ \begin{bmatrix} 4 \\ -2 \end{bmatrix} \right\}$$

For problems C30–C33, row-reduce the matrix without the aid of a calculator, indicating the row operations you are using at each step using the notation of Definition RO.

C30 (Robert Beezer)

$$\begin{bmatrix} 2 & 1 & 5 & 10 \\ 1 & -3 & -1 & -2 \\ 4 & -2 & 6 & 12 \end{bmatrix}$$

Solution (Robert Beezer)

$$\begin{bmatrix} 2 & 1 & 5 & 10 \\ 1 & -3 & -1 & -2 \\ 4 & -2 & 6 & 12 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -3 & -1 & -2 \\ 2 & 1 & 5 & 10 \\ 4 & -2 & 6 & 12 \end{bmatrix}$$

$$\xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & -3 & -1 & -2 \\ 0 & 7 & 7 & 14 \\ 4 & -2 & 6 & 12 \end{bmatrix} \xrightarrow{-4R_1+R_3} \begin{bmatrix} 1 & -3 & -1 & -2 \\ 0 & 7 & 7 & 14 \\ 0 & 10 & 10 & 20 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{7}R_2} \begin{bmatrix} 1 & -3 & -1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 10 & 10 & 20 \end{bmatrix} \xrightarrow{3R_2+R_1} \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 10 & 10 & 20 \end{bmatrix}$$

$$\xrightarrow{-10R_2+R_3} \begin{bmatrix} \boxed{1} & 0 & 2 & 4 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

C31 (Robert Beezer)

$$\begin{bmatrix} 1 & 2 & -4 \\ -3 & -1 & -3 \\ -2 & 1 & -7 \end{bmatrix}$$

Solution (Robert Beezer)

$$\begin{bmatrix} 1 & 2 & -4 \\ -3 & -1 & -3 \\ -2 & 1 & -7 \end{bmatrix} \xrightarrow{3R_1+R_2} \begin{bmatrix} 1 & 2 & -4 \\ 0 & 5 & -15 \\ -2 & 1 & -7 \end{bmatrix}$$

$$\xrightarrow{2R_1+R_3} \begin{bmatrix} 1 & 2 & -4 \\ 0 & 5 & -15 \\ 0 & 5 & -15 \end{bmatrix} \xrightarrow{\frac{1}{5}R_2} \begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & -3 \\ 0 & 5 & -15 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 5 & -15 \end{bmatrix} \xrightarrow{-5R_2+R_3} \begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

C32 (Robert Beezer)

$$\begin{bmatrix} 1 & 1 & 1 \\ -4 & -3 & -2 \\ 3 & 2 & 1 \end{bmatrix}$$

Solution (Robert Beezer) Following the algorithm of Theorem REMEF, and working to create pivot columns from left to right, we have

$$\begin{bmatrix} 1 & 1 & 1 \\ -4 & -3 & -2 \\ 3 & 2 & 1 \end{bmatrix} \xrightarrow{4R_1+R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} \xrightarrow{-3R_1+R_3} \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\xrightarrow{-1R_2+R_1} \begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{1R_2+R_3} \begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

C33 (Robert Beezer)

$$\begin{bmatrix} 1 & 2 & -1 & -1 \\ 2 & 4 & -1 & 4 \\ -1 & -2 & 3 & 5 \end{bmatrix}$$

Solution (Robert Beezer) Following the algorithm of Theorem REMEF, and working to create pivot columns from left to right, we have

$$\begin{bmatrix}1 & 2 & -1 & -1\\ 2 & 4 & -1 & 4\\ -1 & -2 & 3 & 5\end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix}1 & 2 & -1 & -1\\ 0 & 0 & 1 & 6\\ -1 & -2 & 3 & 5\end{bmatrix}$$

$$\xrightarrow{1R_1+R_3} \begin{bmatrix}\boxed{1} & 2 & -1 & -1\\ 0 & 0 & 1 & 6\\ 0 & 0 & 2 & 4\end{bmatrix} \xrightarrow{1R_2+R_1} \begin{bmatrix}\boxed{1} & 2 & 0 & 5\\ 0 & 0 & 1 & 6\\ 0 & 0 & 2 & 4\end{bmatrix}$$

$$\xrightarrow{-2R_2+R_3} \begin{bmatrix}\boxed{1} & 2 & 0 & 5\\ 0 & 0 & \boxed{1} & 6\\ 0 & 0 & 0 & -8\end{bmatrix} \xrightarrow{-\frac{1}{8}R_3} \begin{bmatrix}\boxed{1} & 2 & 0 & 5\\ 0 & 0 & \boxed{1} & 6\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\xrightarrow{-6R_3+R_2} \begin{bmatrix}\boxed{1} & 2 & 0 & 5\\ 0 & 0 & \boxed{1} & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \xrightarrow{-5R_3+R_1} \begin{bmatrix}\boxed{1} & 2 & 0 & 0\\ 0 & 0 & \boxed{1} & 0\\ 0 & 0 & 0 & \boxed{1}\end{bmatrix}$$

M40 (Robert Beezer) Consider the two 3×4 matrices below

$$B = \begin{bmatrix}1 & 3 & -2 & 2\\ -1 & -2 & -1 & -1\\ -1 & -5 & 8 & -3\end{bmatrix} \qquad C = \begin{bmatrix}1 & 2 & 1 & 2\\ 1 & 1 & 4 & 0\\ -1 & -1 & -4 & 1\end{bmatrix}$$

1. Row-reduce each matrix and determine that the reduced row-echelon forms of B and C are identical. From this argue that B and C are row-equivalent.
2. In the proof of Theorem RREFU, we begin by arguing that entries of row-equivalent matrices are related by way of certain scalars and sums. In this example, we would write that entries of B from row i that are in column j are linearly related to the entries of C in column j from all three rows
$$[B]_{ij} = \delta_{i1}\,[C]_{1j} + \delta_{i2}\,[C]_{2j} + \delta_{i3}\,[C]_{3j} \qquad 1 \le j \le 4$$
For each $1 \le i \le 3$ find the corresponding three scalars in this relationship. So your answer will be nine scalars, determined three at a time.

Solution (Robert Beezer)

1. Let R be the common reduced row-echelon form of B and C. A sequence of row operations converts B to R and a second sequence of row operations converts C to R. If we "reverse" the second sequence's order, and reverse each individual row operation (see Exercise RREF.T10) then we can begin with B, convert to R with the first sequence, and then convert to C with the reversed sequence. Satisfying Definition REM we can say B and C are row-equivalent matrices.
2. We will work this carefully for the first row of B and just give the solution for the next two rows. For row 1 of B take $i = 1$ and we have
$$[B]_{1j} = \delta_{11}\,[C]_{1j} + \delta_{12}\,[C]_{2j} + \delta_{13}\,[C]_{3j} \qquad 1 \le j \le 4$$
If we substitute the four values for j we arrive at four linear equations in the three unknowns $\delta_{11}, \delta_{12}, \delta_{13}$,
$$\begin{aligned}
(j=1)\quad & [B]_{11} = \delta_{11}\,[C]_{11} + \delta_{12}\,[C]_{21} + \delta_{13}\,[C]_{31} & \Rightarrow\quad & 1 = \delta_{11}(1) + \delta_{12}(1) + \delta_{13}(-1)\\
(j=2)\quad & [B]_{12} = \delta_{11}\,[C]_{12} + \delta_{12}\,[C]_{22} + \delta_{13}\,[C]_{32} & \Rightarrow\quad & 3 = \delta_{11}(2) + \delta_{12}(1) + \delta_{13}(-1)\\
(j=3)\quad & [B]_{13} = \delta_{11}\,[C]_{13} + \delta_{12}\,[C]_{23} + \delta_{13}\,[C]_{33} & \Rightarrow\quad & -2 = \delta_{11}(1) + \delta_{12}(4) + \delta_{13}(-4)\\
(j=4)\quad & [B]_{14} = \delta_{11}\,[C]_{14} + \delta_{12}\,[C]_{24} + \delta_{13}\,[C]_{34} & \Rightarrow\quad & 2 = \delta_{11}(2) + \delta_{12}(0) + \delta_{13}(1)
\end{aligned}$$
We form the augmented matrix of this system and row-reduce to find the solutions,
$$\begin{bmatrix}1 & 1 & -1 & 1\\ 2 & 1 & -1 & 3\\ 1 & 4 & -4 & -2\\ 2 & 0 & 1 & 2\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & 0 & 2\\ 0 & \boxed{1} & 0 & -3\\ 0 & 0 & \boxed{1} & -2\\ 0 & 0 & 0 & 0\end{bmatrix}$$

So the unique solution is $\delta_{11} = 2$, $\delta_{12} = -3$, $\delta_{13} = -2$. Entirely similar work will lead you to

$$\delta_{21} = -1 \qquad \delta_{22} = 1 \qquad \delta_{23} = 1$$

and

$$\delta_{31} = -4 \qquad \delta_{32} = 8 \qquad \delta_{33} = 5$$

M45 (Chris Black) You keep a number of lizards, mice and peacocks as pets. There are a total of 108 legs and 30 tails in your menagerie. You have twice as many mice as lizards. How many of each creature do you have?

Solution (Chris Black) Let l, m, p denote the number of lizards, mice and peacocks. Then the statements from the problem yield the equations:

$$\begin{aligned} 4l + 4m + 2p &= 108 \\ l + m + p &= 30 \\ 2l - m &= 0 \end{aligned}$$

We form the augmented matrix for this system and row-reduce

$$\begin{bmatrix} 4 & 4 & 2 & 108 \\ 1 & 1 & 1 & 30 \\ 2 & -1 & 0 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 8 \\ 0 & \boxed{1} & 0 & 16 \\ 0 & 0 & \boxed{1} & 6 \end{bmatrix}$$

From the row-reduced matrix, we see that we have an equivalent system $l = 8$, $m = 16$, and $p = 6$, which means that you have 8 lizards, 16 mice and 6 peacocks.

M50 (Robert Beezer) A parking lot has 66 vehicles (cars, trucks, motorcycles and bicycles) in it. There are four times as many cars as trucks. The total number of tires (4 per car or truck, 2 per motorcycle or bicycle) is 252. How many cars are there? How many bicycles?

Solution (Robert Beezer) Let c, t, m, b denote the number of cars, trucks, motorcycles, and bicycles. Then the statements from the problem yield the equations:

$$\begin{aligned} c + t + m + b &= 66 \\ c - 4t &= 0 \\ 4c + 4t + 2m + 2b &= 252 \end{aligned}$$

We form the augmented matrix for this system and row-reduce

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 66 \\ 1 & -4 & 0 & 0 & 0 \\ 4 & 4 & 2 & 2 & 252 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 48 \\ 0 & \boxed{1} & 0 & 0 & 12 \\ 0 & 0 & \boxed{1} & 1 & 6 \end{bmatrix}$$

The first row of the matrix represents the equation $c = 48$, so there are 48 cars. The second row of the matrix represents the equation $t = 12$, so there are 12 trucks. The third row of the matrix represents the equation $m + b = 6$ so there are anywhere from 0 to 6 bicycles. We can also say that b is a free variable, but the context of the problem limits it to 7 integer values since you cannot have a negative number of motorcycles.

T10 (Robert Beezer) Prove that each of the three row operations (Definition RO) is reversible. More precisely, if the matrix B is obtained from A by application of a single row operation, show that there is a single row operation that will transform B back into A.

Solution (Robert Beezer) If we can reverse each row operation individually, then we can reverse a sequence of row operations. The operations that reverse each operation are listed below, using our shorthand notation. Notice how requiring the scalar α to be non-zero makes the second operation reversible.

$$\begin{aligned} R_i \leftrightarrow R_j &\qquad R_i \leftrightarrow R_j \\ \alpha R_i,\, \alpha \neq 0 &\qquad \frac{1}{\alpha} R_i \\ \alpha R_i + R_j &\qquad -\alpha R_i + R_j \end{aligned}$$

T11 (Robert Beezer) Suppose that A, B and C are $m \times n$ matrices. Use the definition of row-equivalence (Definition REM) to prove the following three facts.

1. A is row-equivalent to A.
2. If A is row-equivalent to B, then B is row-equivalent to A.
3. If A is row-equivalent to B, and B is row-equivalent to C, then A is row-equivalent to C.

A relationship that satisfies these three properties is known as an **equivalence relation**, an important idea in the study of various algebras. This is a formal way of saying that a relationship behaves like equality, without requiring the relationship to be as strict as equality itself. We'll see it again in Theorem SER.

T12 (Robert Beezer) Suppose that B is an $m \times n$ matrix in reduced row-echelon form. Build a new, likely smaller, $k \times \ell$ matrix C as follows. Keep any collection of k adjacent rows, $k \leq m$. From these rows, keep columns 1 through ℓ, $\ell \leq n$. Prove that C is in reduced row-echelon form.

T13 (Robert Beezer) Generalize Exercise RREF.T12 by just keeping any k rows, and not requiring the rows to be adjacent. Prove that any such matrix C is in reduced row-echelon form.

Section TSS
Types of Solution Sets

C10 (Robert Beezer) In the spirit of Example ISSI, describe the infinite solution set for Archetype J.

For Exercises C21–C28, find the solution set of the system of linear equations. Give the values of n and r, and interpret your answers in light of the theorems of this section.

C21 (Chris Black)

$$\begin{aligned} x_1 + 4x_2 + 3x_3 - x_4 &= 5 \\ x_1 - x_2 + x_3 + 2x_4 &= 6 \\ 4x_1 + x_2 + 6x_3 + 5x_4 &= 9 \end{aligned}$$

Solution (Chris Black) The augmented matrix for the given linear system and its row-reduced form are:

$$\begin{bmatrix} 1 & 4 & 3 & -1 & 5 \\ 1 & -1 & 1 & 2 & 6 \\ 4 & 1 & 6 & 5 & 9 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 7/5 & 7/5 & 0 \\ 0 & \boxed{1} & 2/5 & -3/5 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}.$$

For this system, we have $n = 4$ and $r = 3$. However, with a leading 1 in the last column we see that the original system has no solution by Theorem RCLS.

C22 (Chris Black)

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= 3 \\ 2x_1 - 4x_2 + x_3 + x_4 &= 2 \\ x_1 - 2x_2 - 2x_3 + 3x_4 &= 1 \end{aligned}$$

Solution (Chris Black) The augmented matrix for the given linear system and its row-reduced form are:

$$\begin{bmatrix} 1 & -2 & 1 & -1 & 3 \\ 2 & -4 & 1 & 1 & 2 \\ 1 & -2 & -2 & 3 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & -2 & 0 & 0 & 3 \\ 0 & 0 & \boxed{1} & 0 & -2 \\ 0 & 0 & 0 & \boxed{1} & -2 \end{bmatrix}.$$

Thus, we see we have an equivalent system for any scalar x_2:

$$\begin{aligned} x_1 &= 3 + 2x_2 \\ x_3 &= -2 \end{aligned}$$

$$x_4 = -2.$$

For this system, $n = 4$ and $r = 3$. Since it is a consistent system by Theorem RCLS, Theorem CSRN guarantees an infinite number of solutions.

C23 (Chris Black)

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= 3 \\ x_1 + x_2 + x_3 - x_4 &= 1 \\ x_1 \quad + x_3 - x_4 &= 2 \end{aligned}$$

Solution (Chris Black) The augmented matrix for the given linear system and its row-reduced form are:

$$\begin{bmatrix} 1 & -2 & 1 & -1 & 3 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1 & -1 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}.$$

For this system, we have $n = 4$ and $r = 3$. However, with a leading 1 in the last column we see that the original system has no solution by Theorem RCLS.

C24 (Chris Black)

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= 2 \\ x_1 + x_2 + x_3 - x_4 &= 2 \\ x_1 \quad + x_3 - x_4 &= 2 \end{aligned}$$

Solution (Chris Black) The augmented matrix for the given linear system and its row-reduced form are:

$$\begin{bmatrix} 1 & -2 & 1 & -1 & 2 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 0 & 1 & -1 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1 & -1 & 2 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, we see that an equivalent system is

$$\begin{aligned} x_1 &= 2 - x_3 + x_4 \\ x_2 &= 0, \end{aligned}$$

and the solution set is $\left\{ \begin{bmatrix} 2 - x_3 + x_4 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} \middle| x_3, x_4 \in \mathbb{C} \right\}$. For this system, $n = 4$ and $r = 2$. Since it is a consistent system by Theorem RCLS, Theorem CSRN guarantees an infinite number of solutions.

C25 (Chris Black)

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 2 \\ 3x_1 + x_2 + x_3 &= 4 \\ x_2 + 2x_3 &= 6 \end{aligned}$$

Solution (Chris Black) The augmented matrix for the given linear system and its row-reduced form are:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & -1 & 1 & 2 \\ 3 & 1 & 1 & 4 \\ 0 & 1 & 2 & 6 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}.$$

Since $n = 3$ and $r = 4 = n + 1$, Theorem ISRN guarantees that the system is inconsistent. Thus, we see that the given system has no solution.

C26 (Chris Black)

$$x_1 + 2x_2 + 3x_3 = 1$$

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 2 \\ 3x_1 + x_2 + x_3 &= 4 \\ 5x_2 + 2x_3 &= 1 \end{aligned}$$

Solution (Chris Black) The augmented matrix for the given linear system and its row-reduced form are:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & -1 & 1 & 2 \\ 3 & 1 & 1 & 4 \\ 0 & 5 & 2 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 4/3 \\ 0 & \boxed{1} & 0 & 1/3 \\ 0 & 0 & \boxed{1} & -1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $r = n = 3$ and the system is consistent by Theorem RCLS, Theorem CSRN guarantees a unique solution, which is

$$\begin{aligned} x_1 &= 4/3 \\ x_2 &= 1/3 \\ x_3 &= -1/3. \end{aligned}$$

C27 (Chris Black)

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 2 \\ x_1 - 8x_2 - 7x_3 &= 1 \\ x_2 + x_3 &= 0 \end{aligned}$$

Solution (Chris Black) The augmented matrix for the given linear system and its row-reduced form are:

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & -1 & 1 & 2 \\ 1 & -8 & -7 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For this system, we have $n = 3$ and $r = 3$. However, with a leading 1 in the last column we see that the original system has no solution by Theorem RCLS.

C28 (Chris Black)

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 2 \\ x_1 - 8x_2 - 7x_3 &= 1 \\ x_2 + x_3 &= 0 \end{aligned}$$

Solution (Chris Black) The augmented matrix for the given linear system and its row-reduced form are:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & -1 & 1 & 2 \\ 1 & -8 & -7 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1 & 1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For this system, $n = 3$ and $r = 2$. Since it is a consistent system by Theorem RCLS, Theorem CSRN guarantees an infinite number of solutions. An equivalent system is

$$\begin{aligned} x_1 &= 1 - x_3 \\ x_2 &= -x_3, \end{aligned}$$

where x_3 is any scalar. So we can express the solution set as

$$\left\{ \begin{bmatrix} 1 - x_3 \\ -x_3 \\ x_3 \end{bmatrix} \middle| x_3 \in \mathbb{C} \right\}$$

M45 (Robert Beezer) The details for Archetype J include several sample solutions. Verify that one of these solutions is correct (any one, but just one). Based *only* on this evidence, and especially without doing any row operations, explain how you know this system of linear equations has infinitely many solutions.

Solution (Robert Beezer) Demonstrate that the system is consistent by verifying any one of the four sample solutions provided. Then because $n = 9 > 6 = m$, Theorem CMVEI gives us the conclusion that the system has infinitely many solutions. Notice that we only know the system will have *at least* $9 - 6 = 3$ free variables, but very well could have more. We do not know know that $r = 6$, only that $r \leq 6$.

M46 (Manley Perkel) Consider Archetype J, and specifically the row-reduced version of the augmented matrix of the system of equations, denoted as B here, and the values of r, D and F immediately following. Determine the values of the entries

$$[B]_{1,d_1} \quad [B]_{3,d_3} \quad [B]_{1,d_3} \quad [B]_{3,d_1} \quad [B]_{d_1,1} \quad [B]_{d_3,3} \quad [B]_{d_1,3} \quad [B]_{d_3,1} \quad [B]_{1,f_1} \quad [B]_{3,f_1}$$

(See Exercise TSS.M70 for a generalization.)

For Exercises M51–M57 say **as much as possible** about each system's solution set. Be sure to make it clear which theorems you are using to reach your conclusions.

M51 (Robert Beezer) A consistent system of 8 equations in 6 variables.

Solution (Robert Beezer) Consistent means there is at least one solution (Definition CS). It will have either a unique solution or infinitely many solutions (Theorem PSSLS).

M52 (Robert Beezer) A consistent system of 6 equations in 8 variables.

Solution (Robert Beezer) With 6 rows in the augmented matrix, the row-reduced version will have $r \leq 6$. Since the system is consistent, apply Theorem CSRN to see that $n - r \geq 2$ implies infinitely many solutions.

M53 (Robert Beezer) A system of 5 equations in 9 variables.

Solution (Robert Beezer) The system could be inconsistent. If it is consistent, then because it has more variables than equations Theorem CMVEI implies that there would be infinitely many solutions. So, of all the possibilities in Theorem PSSLS, only the case of a unique solution can be ruled out.

M54 (Robert Beezer) A system with 12 equations in 35 variables.

Solution (Robert Beezer) The system could be inconsistent. If it is consistent, then Theorem CMVEI tells us the solution set will be infinite. So we can be certain that there is not a unique solution.

M56 (Robert Beezer) A system with 6 equations in 12 variables.

Solution (Robert Beezer) The system could be inconsistent. If it is consistent, and since $12 > 6$, then Theorem CMVEI says we will have infinitely many solutions. So there are two possibilities. Theorem PSSLS allows to state equivalently that a unique solution is an impossibility.

M57 (Robert Beezer) A system with 8 equations and 6 variables. The reduced row-echelon form of the augmented matrix of the system has 7 pivot columns.

Solution (Robert Beezer) 7 pivot columns implies that there are $r = 7$ nonzero rows (so row 8 is all zeros in the reduced row-echelon form). Then $n + 1 = 6 + 1 = 7 = r$ and Theorem ISRN allows to conclude that the system is inconsistent.

M60 (Robert Beezer) Without doing any computations, and without examining any solutions, say as much as possible about the form of the solution set for each archetype that is a system of equations.

Archetype A, Archetype B, Archetype C, Archetype D, Archetype E, Archetype F, Archetype G, Archetype H, Archetype I, Archetype J

M70 (Manley Perkel) Suppose that B is a matrix in reduced row-echelon form that is equivalent to the augmented matrix of a system of equations with m equations in n variables. Let r, D and F be as defined in Definition RREF. What can you conclude, in general, about the following entries?

$$[B]_{1,d_1} \quad [B]_{3,d_3} \quad [B]_{1,d_3} \quad [B]_{3,d_1} \quad [B]_{d_1,1} \quad [B]_{d_3,3} \quad [B]_{d_1,3} \quad [B]_{d_3,1} \quad [B]_{1,f_1} \quad [B]_{3,f_1}$$

If you cannot conclude anything about an entry, then say so. (See Exercise TSS.M46.)

T10 (Robert Beezer) An inconsistent system may have $r > n$. If we try (incorrectly!) to apply Theorem FVCS to such a system, how many free variables would we discover?

Solution (Robert Beezer) Theorem FVCS will indicate a negative number of free variables, but we can say even more. If $r > n$, then the only possibility is that $r = n+1$, and then we compute $n-r = n-(n+1) = -1$ free variables.

T20 (Manley Perkel) Suppose that B is a matrix in reduced row-echelon form that is equivalent to the augmented matrix of a system of equations with m equations in n variables. Let r, D and F be as defined in Definition RREF. Prove that $d_k \geq k$ for all $1 \leq k \leq r$. Then suppose that $r \geq 2$ and $1 \leq k < \ell \leq r$ and determine what can you conclude, in general, about the following entries.

$$[B]_{k,d_k} \quad [B]_{k,d_\ell} \quad [B]_{\ell,d_k} \quad [B]_{d_k,k} \quad [B]_{d_k,\ell} \quad [B]_{d_\ell,k} \quad [B]_{d_k,f_\ell} \quad [B]_{d_\ell,f_k}$$

If you cannot conclude anything about an entry, then say so. (See Exercise TSS.M46 and Exercise TSS.M70.)

T40 (Robert Beezer) Suppose that the coefficient matrix of a consistent system of linear equations has two columns that are identical. Prove that the system has infinitely many solutions.

Solution (Robert Beezer) Since the system is consistent, we know there is either a unique solution, or infinitely many solutions (Theorem PSSLS). If we perform row operations (Definition RO) on the augmented matrix of the system, the two equal columns of the coefficient matrix will suffer the same fate, and remain equal in the final reduced row-echelon form. Suppose both of these columns are pivot columns (Definition RREF). Then there is single row containing the two leading 1's of the two pivot columns, a violation of reduced row-echelon form (Definition RREF). So at least one of these columns is not a pivot column, and the column index indicates a free variable in the description of the solution set (Definition IDV). With a free variable, we arrive at an infinite solution set (Theorem FVCS).

T41 (Robert Beezer) Consider the system of linear equations $\mathcal{LS}(A, \mathbf{b})$, and suppose that every element of the vector of constants $\mathbf{b}$ is a common multiple of the corresponding element of a certain column of A. More precisely, there is a complex number α, and a column index j, such that $[\mathbf{b}]_i = \alpha\,[A]_{ij}$ for all i. Prove that the system is consistent.

Solution (Robert Beezer) The condition about the multiple of the column of constants will allow you to show that the following values form a solution of the system $\mathcal{LS}(A, \mathbf{b})$,

$$x_1 = 0 \quad x_2 = 0 \quad \dots \quad x_{j-1} = 0 \quad x_j = \alpha \quad x_{j+1} = 0 \quad \dots \quad x_{n-1} = 0 \quad x_n = 0$$

With one solution of the system known, we can say the system is consistent (Definition CS).

A more involved proof can be built using Theorem RCLS. Begin by proving that each of the three row operations (Definition RO) will convert the augmented matrix of the system into another matrix where column j is α times the entry of the same row in the last column. In other words, the "column multiple property" is preserved under row operations. These proofs will get successively more involved as you work through the three operations.

Now construct a proof by contradiction (Proof Technique CD), by supposing that the system is inconsistent. Then the last column of the reduced row-echelon form of the augmented matrix is a pivot column (Theorem RCLS). Then column j must have a zero in the same row as the leading 1 of the final column. But the "column multiple property" implies that there is an α in column j in the same row as the leading 1. So $\alpha = 0$. By hypothesis, then the vector of constants is the zero vector. However, if we began with a final column of zeros, row operations would never have created a leading 1 in the final column. This contradicts the final column being a pivot column, and therefore the system cannot be inconsistent.

Section HSE
Homogeneous Systems of Equations

C10 (Robert Beezer) Each Archetype (Archetypes) that is a system of equations has a corresponding homogeneous system with the same coefficient matrix. Compute the set of solutions for each. Notice that these solution sets are the null spaces of the coefficient matrices.

Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J

C20 (Robert Beezer) Archetype K and Archetype L are simply 5×5 matrices (i.e. they are not systems of equations). Compute the null space of each matrix.

For Exercises C21-C23, solve the given homogeneous linear system. Compare your results to the results of the corresponding exercise in Section TSS.

C21 (Chris Black)

$$\begin{aligned} x_1 + 4x_2 + 3x_3 - x_4 &= 0 \\ x_1 - x_2 + x_3 + 2x_4 &= 0 \\ 4x_1 + x_2 + 6x_3 + 5x_4 &= 0 \end{aligned}$$

Solution (Chris Black) The augmented matrix for the given linear system and its row-reduced form are:

$$\begin{bmatrix} 1 & 4 & 3 & -1 & 0 \\ 1 & -1 & 1 & 2 & 0 \\ 4 & 1 & 6 & 5 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 7/5 & 7/5 & 0 \\ 0 & \boxed{1} & 2/5 & -3/5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, we see that the system is consistent (as predicted by Theorem HSC) and has an infinite number of solutions (as predicted by Theorem HMVEI). With suitable choices of x_3 and x_4, each solution can be written as

$$\begin{bmatrix} -\frac{7}{5}x_3 - \frac{7}{5}x_4 \\ -\frac{2}{5}x_3 + \frac{3}{5}x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

C22 (Chris Black)

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= 0 \\ 2x_1 - 4x_2 + x_3 + x_4 &= 0 \\ x_1 - 2x_2 - 2x_3 + 3x_4 &= 0 \end{aligned}$$

Solution (Chris Black) The augmented matrix for the given linear system and its row-reduced form are:

$$\begin{bmatrix} 1 & -2 & 1 & -1 & 0 \\ 2 & -4 & 1 & 1 & 0 \\ 1 & -2 & -2 & 3 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & -2 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{bmatrix}.$$

Thus, we see that the system is consistent (as predicted by Theorem HSC) and has an infinite number of solutions (as predicted by Theorem HMVEI). With a suitable choice of x_2, each solution can be written as

$$\begin{bmatrix} 2x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix}$$

C23 (Chris Black)

$$x_1 - 2x_2 + x_3 - x_4 = 0$$

$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 0 \\ x_1 \quad + x_3 - x_4 &= 0 \end{aligned}$$

Solution (Chris Black) The augmented matrix for the given linear system and its row-reduced form are:

$$\begin{bmatrix} 1 & -2 & 1 & -1 & 0 \\ 1 & 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1 & -1 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, we see that the system is consistent (as predicted by Theorem HSC) and has an infinite number of solutions (as predicted by Theorem HMVEI). With suitable choices of x_3 and x_4, each solution can be written as

$$\begin{bmatrix} -x_3 + x_4 \\ 0 \\ x_3 \\ x_4 \end{bmatrix}$$

For Exercises C25-C27, solve the given homogeneous linear system. Compare your results to the results of the corresponding exercise in Section TSS.

C25 (Chris Black)

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 0 \\ 3x_1 + x_2 + x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

Solution (Chris Black) The augmented matrix for the given linear system and its row-reduced form are:

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & -1 & 1 & 0 \\ 3 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

An homogeneous system is always consistent (Theorem HSC) and with $n = r = 3$ an application of Theorem FVCS yields zero free variables. Thus the only solution to the given system is the trivial solution, $\mathbf{x} = \mathbf{0}$.

C26 (Chris Black)

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 0 \\ 3x_1 + x_2 + x_3 &= 0 \\ 5x_2 + 2x_3 &= 0 \end{aligned}$$

Solution (Chris Black) The augmented matrix for the given linear system and its row-reduced form are:

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & -1 & 1 & 0 \\ 3 & 1 & 1 & 0 \\ 0 & 5 & 2 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

An homogeneous system is always consistent (Theorem HSC) and with $n = r = 3$ an application of Theorem FVCS yields zero free variables. Thus the only solution to the given system is the trivial solution, $\mathbf{x} = \mathbf{0}$.

C27 (Chris Black)

$$x_1 + 2x_2 + 3x_3 = 0$$

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 0 \\ x_1 - 8x_2 - 7x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

Solution (Chris Black) The augmented matrix for the given linear system and its row-reduced form are:

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & -1 & 1 & 0 \\ 1 & -8 & -7 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

An homogeneous system is always consistent (Theorem HSC) and with $n = 3$, $r = 2$ an application of Theorem FVCS yields one free variable. With a suitable choice of x_3 each solution can be written in the form

$$\begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix}$$

C30 (Robert Beezer) Compute the null space of the matrix A, $\mathcal{N}(A)$.

$$A = \begin{bmatrix} 2 & 4 & 1 & 3 & 8 \\ -1 & -2 & -1 & -1 & 1 \\ 2 & 4 & 0 & -3 & 4 \\ 2 & 4 & -1 & -7 & 4 \end{bmatrix}$$

Solution (Robert Beezer) Definition NSM tells us that the null space of A is the solution set to the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. The augmented matrix of this system is

$$\begin{bmatrix} 2 & 4 & 1 & 3 & 8 & 0 \\ -1 & -2 & -1 & -1 & 1 & 0 \\ 2 & 4 & 0 & -3 & 4 & 0 \\ 2 & 4 & -1 & -7 & 4 & 0 \end{bmatrix}$$

To solve the system, we row-reduce the augmented matrix and obtain,

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 0 & 5 & 0 \\ 0 & 0 & \boxed{1} & 0 & -8 & 0 \\ 0 & 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix represents a system with equations having three dependent variables (x_1, x_3, and x_4) and two independent variables (x_2 and x_5). These equations rearrange to

$$x_1 = -2x_2 - 5x_5 \qquad x_3 = 8x_5 \qquad x_4 = -2x_5$$

So we can write the solution set (which is the requested null space) as

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} -2x_2 - 5x_5 \\ x_2 \\ 8x_5 \\ -2x_5 \\ x_5 \end{bmatrix} \middle| \, x_2, x_5 \in \mathbb{C} \right\}$$

C31 (Robert Beezer) Find the null space of the matrix B, $\mathcal{N}(B)$.

$$B = \begin{bmatrix} -6 & 4 & -36 & 6 \\ 2 & -1 & 10 & -1 \\ -3 & 2 & -18 & 3 \end{bmatrix}$$

Solution (Robert Beezer) We form the augmented matrix of the homogeneous system $\mathcal{LS}(B, \mathbf{0})$ and row-reduce the matrix,

$$\begin{bmatrix} -6 & 4 & -36 & 6 & 0 \\ 2 & -1 & 10 & -1 & 0 \\ -3 & 2 & -18 & 3 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2 & 1 & 0 \\ 0 & \boxed{1} & -6 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We knew ahead of time that this system would be consistent (Theorem HSC), but we can now see there are $n-r = 4-2 = 2$ free variables, namely x_3 and x_4 (Theorem FVCS). Based on this analysis, we can rearrange the equations associated with each nonzero row of the reduced row-echelon form into an expression for the lone dependent variable as a function of the free variables. We arrive at the solution set to the homogeneous system, which is the null space of the matrix by Definition NSM,

$$\mathcal{N}(B) = \left\{ \begin{bmatrix} -2x_3 - x_4 \\ 6x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} \middle|\, x_3,\, x_4 \in \mathbb{C} \right\}$$

M45 (Robert Beezer) Without doing any computations, and without examining any solutions, say as much as possible about the form of the solution set for corresponding homogeneous system of equations of each archetype that is a system of equations.

Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J

For Exercises M50–M52 say **as much as possible** about each system's solution set. Be sure to make it clear which theorems you are using to reach your conclusions.

M50 (Robert Beezer) A homogeneous system of 8 equations in 8 variables.

Solution (Robert Beezer) Since the system is homogeneous, we know it has the trivial solution (Theorem HSC). We cannot say anymore based on the information provided, except to say that there is either a unique solution or infinitely many solutions (Theorem PSSLS). See Archetype A and Archetype B to understand the possibilities.

M51 (Robert Beezer) A homogeneous system of 8 equations in 9 variables.

Solution (Robert Beezer) Since there are more variables than equations, Theorem HMVEI applies and tells us that the solution set is infinite. From the proof of Theorem HSC we know that the zero vector is one solution.

M52 (Robert Beezer) A homogeneous system of 8 equations in 7 variables.

Solution (Robert Beezer) By Theorem HSC, we know the system is consistent because the zero vector is always a solution of a homogeneous system. There is no more that we can say, since both a unique solution and infinitely many solutions are possibilities.

T10 (Martin Jackson) Prove or disprove: A system of linear equations is homogeneous if and only if the system has the zero vector as a solution.

Solution (Robert Beezer) This is a true statement. A proof is:

($\Rightarrow$) Suppose we have a homogeneous system $\mathcal{LS}(A,\, \mathbf{0})$. Then by substituting the scalar zero for each variable, we arrive at true statements for each equation. So the zero vector is a solution. This is the content of Theorem HSC.

($\Leftarrow$) Suppose now that we have a generic (i.e. not necessarily homogeneous) system of equations, $\mathcal{LS}(A,\, \mathbf{b})$ that has the zero vector as a solution. Upon substituting this solution into the system, we discover that each component of $\mathbf{b}$ must also be zero. So $\mathbf{b} = \mathbf{0}$.

T11 (Robert Beezer) Suppose that two systems of linear equations are equivalent. Prove that if the first system is homogeneous, then the second system is homogeneous. Notice that this will allow us to conclude that two equivalent systems are either both homogeneous or both not homogeneous.

Solution (Robert Beezer) If the first system is homogeneous, then the zero vector is in the solution set of the system. (See the proof of Theorem HSC.)

Since the two systems are equivalent, they have equal solutions sets (Definition ESYS). So the zero vector is in the solution set of the second system. By Exercise HSE.T10 the presence of the zero vector in the solution set implies that the system is homogeneous.

So if any one of two equivalent systems is homogeneous, then they both are homogeneous.

T12 (Ivan Kessler) Give an alternate proof of Theorem HSC that uses Theorem RCLS.

T20 (Robert Beezer) Consider the homogeneous system of linear equations $\mathcal{LS}(A,\,\mathbf{0})$, and suppose that $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix}$ is one solution to the system of equations. Prove that $\mathbf{v} = \begin{bmatrix} 4u_1 \\ 4u_2 \\ 4u_3 \\ \vdots \\ 4u_n \end{bmatrix}$ is also a solution to $\mathcal{LS}(A,\,\mathbf{0})$.

Solution (Robert Beezer) Suppose that a single equation from this system (the i-th one) has the form,

$$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n = 0$$

Evaluate the left-hand side of this equation with the components of the proposed solution vector $\mathbf{v}$,

$$\begin{aligned} &a_{i1}(4u_1) + a_{i2}(4u_2) + a_{i3}(4u_3) + \cdots + a_{in}(4u_n) \\ &\quad = 4a_{i1}u_1 + 4a_{i2}u_2 + 4a_{i3}u_3 + \cdots + 4a_{in}u_n && \text{Commutativity} \\ &\quad = 4(a_{i1}u_1 + a_{i2}u_2 + a_{i3}u_3 + \cdots + a_{in}u_n) && \text{Distributivity} \\ &\quad = 4(0) && \mathbf{u} \text{ solution to } \mathcal{LS}(A,\,\mathbf{0}) \\ &\quad = 0 \end{aligned}$$

So $\mathbf{v}$ makes each equation true, and so is a solution to the system.

Notice that this result is not true if we change $\mathcal{LS}(A,\,\mathbf{0})$ from a homogeneous system to a non-homogeneous system. Can you create an example of a (non-homogeneous) system with a solution $\mathbf{u}$ such that $\mathbf{v}$ is not a solution?

Section NM
Nonsingular Matrices

In Exercises C30–C33 determine if the matrix is nonsingular or singular. Give reasons for your answer.

C30 (Robert Beezer)

$$\begin{bmatrix} -3 & 1 & 2 & 8 \\ 2 & 0 & 3 & 4 \\ 1 & 2 & 7 & -4 \\ 5 & -1 & 2 & 0 \end{bmatrix}$$

Solution (Robert Beezer) The matrix row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

which is the 4×4 identity matrix. By Theorem NMRRI the original matrix must be nonsingular.

C31 (Robert Beezer)

$$\begin{bmatrix} 2 & 3 & 1 & 4 \\ 1 & 1 & 1 & 0 \\ -1 & 2 & 3 & 5 \\ 1 & 2 & 1 & 3 \end{bmatrix}$$

Solution (Robert Beezer) Row-reducing the matrix yields,

$$\begin{bmatrix} \boxed{1} & 0 & 0 & -2 \\ 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & \boxed{1} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this is not the 4×4 identity matrix, Theorem NMRRI tells us the matrix is singular.

C32 (Robert Beezer)

$$\begin{bmatrix} 9 & 3 & 2 & 4 \\ 5 & -6 & 1 & 3 \\ 4 & 1 & 3 & -5 \end{bmatrix}$$

Solution (Robert Beezer) The matrix is not square, so neither term is applicable. See Definition NM, which is stated for just square matrices.

C33 (Robert Beezer)

$$\begin{bmatrix} -1 & 2 & 0 & 3 \\ 1 & -3 & -2 & 4 \\ -2 & 0 & 4 & 3 \\ -3 & 1 & -2 & 3 \end{bmatrix}$$

Solution (Robert Beezer) Theorem NMRRI tells us we can answer this question by simply row-reducing the matrix. Doing this we obtain,

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

Since the reduced row-echelon form of the matrix is the 4×4 identity matrix I_4, we know that B is nonsingular.

C40 (Robert Beezer) Each of the archetypes below is a system of equations with a square coefficient matrix, or is itself a square matrix. Determine if these matrices are nonsingular, or singular. Comment on the null space of each matrix.

Archetype A, Archetype B, Archetype F, Archetype K, Archetype L

C50 (Robert Beezer) Find the null space of the matrix E below.

$$E = \begin{bmatrix} 2 & 1 & -1 & -9 \\ 2 & 2 & -6 & -6 \\ 1 & 2 & -8 & 0 \\ -1 & 2 & -12 & 12 \end{bmatrix}$$

Solution (Robert Beezer) We form the augmented matrix of the homogeneous system $\mathcal{LS}(E,\,\mathbf{0})$ and row-reduce the matrix,

$$\begin{bmatrix} 2 & 1 & -1 & -9 & 0 \\ 2 & 2 & -6 & -6 & 0 \\ 1 & 2 & -8 & 0 & 0 \\ -1 & 2 & -12 & 12 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2 & -6 & 0 \\ 0 & \boxed{1} & -5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We knew ahead of time that this system would be consistent (Theorem HSC), but we can now see there are $n - r = 4 - 2 = 2$ free variables, namely x_3 and x_4 since $F = \{3, 4, 5\}$ (Theorem FVCS). Based on this analysis, we can rearrange the equations associated with each nonzero row of the reduced row-echelon form into an expression for the lone dependent variable as a function of the free variables. We arrive at the solution set to this homogeneous system, which is the null space of the matrix by Definition NSM,

$$\mathcal{N}(E) = \left\{ \begin{bmatrix} -2x_3 + 6x_4 \\ 5x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} \middle|\, x_3,\, x_4 \in \mathbb{C} \right\}$$

M30 (Robert Beezer) Let A be the coefficient matrix of the system of equations below. Is A nonsingular or singular? Explain what you could infer about the solution set for the system based only on what you have learned about A being singular or nonsingular.

$$\begin{aligned} -x_1 + 5x_2 &= -8 \\ -2x_1 + 5x_2 + 5x_3 + 2x_4 &= 9 \\ -3x_1 - x_2 + 3x_3 + x_4 &= 3 \\ 7x_1 + 6x_2 + 5x_3 + x_4 &= 30 \end{aligned}$$

Solution (Robert Beezer) We row-reduce the coefficient matrix of the system of equations,

$$\begin{bmatrix} -1 & 5 & 0 & 0 \\ -2 & 5 & 5 & 2 \\ -3 & -1 & 3 & 1 \\ 7 & 6 & 5 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

Since the row-reduced version of the coefficient matrix is the 4×4 identity matrix, I_4 (Definition IM byTheorem NMRRI, we know the coefficient matrix is nonsingular. According to Theorem NMUS we know that the system is guaranteed to have a unique solution, based only on the extra information that the coefficient matrix is nonsingular.

For Exercises M51–M52 say **as much as possible** about each system's solution set. Be sure to make it clear which theorems you are using to reach your conclusions.

M51 (Robert Beezer) 6 equations in 6 variables, singular coefficient matrix.

Solution (Robert Beezer) Theorem NMRRI tells us that the coefficient matrix will not row-reduce to the identity matrix. So if we were to row-reduce the augmented matrix of this system of equations, we would not get a unique solution. So by Theorem PSSLS the remaining possibilities are no solutions, or infinitely many.

M52 (Robert Beezer) A system with a nonsingular coefficient matrix, not homogeneous.

Solution (Robert Beezer) Any system with a nonsingular coefficient matrix will have a unique solution by Theorem NMUS. If the system is not homogeneous, the solution cannot be the zero vector (Exercise HSE.T10).

T10 (Robert Beezer) Suppose that A is a square matrix, and B is a matrix in reduced row-echelon form that is row-equivalent to A. Prove that if A is singular, then the last row of B is a zero row.

Solution (Robert Beezer) Let n denote the size of the square matrix A. By Theorem NMRRI the hypothesis that A is singular implies that B is not the identity matrix I_n. If B has n pivot columns, then it would have to be I_n, so B must have fewer than n pivot columns. But the number of nonzero rows in B (r) is equal to the number of pivot columns as well. So the n rows of B have fewer than n nonzero rows, and B must contain at least one zero row. By Definition RREF, this row must be at the bottom of B.

A proof can also be formulated by first forming the contrapositive of the statement (Proof Technique CP) and proving this statement.

T12 (Robert Beezer) Suppose that A is a square matrix. Using the definition of reduced row-echelon form (Definition RREF) carefully, give a proof of the following equivalence: Every column of A is a pivot column if and only if A is the identity matrix (Definition IM).

T30 (Robert Beezer) Suppose that A is a nonsingular matrix and A is row-equivalent to the matrix B. Prove that B is nonsingular.

Solution (Robert Beezer) Since A and B are row-equivalent matrices, consideration of the three row operations (Definition RO) will show that the augmented matrices, $[A \mid \mathbf{0}]$ and $[B \mid \mathbf{0}]$, are also row-equivalent matrices. This says that the two homogeneous systems, $\mathcal{LS}(A, \mathbf{0})$ and $\mathcal{LS}(B, \mathbf{0})$ are equivalent systems. $\mathcal{LS}(A, \mathbf{0})$ has only the zero vector as a solution (Definition NM), thus $\mathcal{LS}(B, \mathbf{0})$ has only the zero vector as a solution. Finally, by Definition NM, we see that B is nonsingular.

Form a similar theorem replacing "nonsingular" by "singular" in both the hypothesis and the conclusion. Prove this new theorem with an approach just like the one above, and/or employ the result about nonsingular matrices in a proof by contradiction.

T31 (Robert Beezer) Suppose that A is a square matrix of size $n \times n$ and that we know there is a *single* vector $\mathbf{b} \in \mathbb{C}^n$ such that the system $\mathcal{LS}(A,\,\mathbf{b})$ has a unique solution. Prove that A is a nonsingular matrix. (Notice that this is very similar to Theorem NMUS, but is not exactly the same.)

Solution (Robert Beezer) Let B be the reduced row-echelon form of the augmented matrix $[\,A\,|\,\mathbf{b}]$. Because the system $\mathcal{LS}(A,\,\mathbf{b})$ is consistent, we know by Theorem RCLS that the last column of B is not a pivot column. Suppose now that $r < n$. Then by Theorem FVCS the system would have infinitely many solutions. From this contradiction, we see that $r = n$ and the first n columns of B are each pivot columns. Then the sequence of row operations that converts $[\,A\,|\,\mathbf{b}]$ to B will also convert A to I_n. Applying Theorem NMRRI we conclude that A is nonsingular.

T90 (Robert Beezer) Provide an alternative for the second half of the proof of Theorem NMUS, without appealing to properties of the reduced row-echelon form of the coefficient matrix. In other words, prove that if A is nonsingular, then $\mathcal{LS}(A,\,\mathbf{b})$ has a unique solution for every choice of the constant vector $\mathbf{b}$. Construct this proof without using Theorem REMEF or Theorem RREFU.

Solution (Robert Beezer) We assume A is nonsingular, and try to solve the system $\mathcal{LS}(A,\,\mathbf{b})$ without making any assumptions about $\mathbf{b}$. To do this we will begin by constructing a new homogeneous linear system of equations that looks very much like the original. Suppose A has size n (why must it be square?) and write the original system as,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3\\
&\vdots\\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n
\end{aligned} \tag{$*$}$$

Form the new, homogeneous system in n equations with $n+1$ variables, by adding a new variable y, whose coefficients are the negatives of the constant terms,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n - b_1y &= 0\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n - b_2y &= 0\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n - b_3y &= 0\\
&\vdots\\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n - b_ny &= 0
\end{aligned} \tag{$**$}$$

Since this is a homogeneous system with more variables than equations ($m = n+1 > n$), Theorem HMVEI says that the system has infinitely many solutions. We will choose one of these solutions, *any* one of these solutions, so long as it is *not* the trivial solution. Write this solution as

$$x_1 = c_1 \qquad x_2 = c_2 \qquad x_3 = c_3 \qquad \ldots \qquad x_n = c_n \qquad y = c_{n+1}$$

We know that at least one value of the c_i is nonzero, but we will now show that in particular $c_{n+1} \neq 0$. We do this using a proof by contradiction (Proof Technique CD). So suppose the c_i form a solution as described, and in addition that $c_{n+1} = 0$. Then we can write the i-th equation of system $(**)$ as,

$$a_{i1}c_1 + a_{i2}c_2 + a_{i3}c_3 + \cdots + a_{in}c_n - b_i(0) = 0$$

which becomes

$$a_{i1}c_1 + a_{i2}c_2 + a_{i3}c_3 + \cdots + a_{in}c_n = 0$$

Since this is true for each i, we have that $x_1 = c_1,\, x_2 = c_2,\, x_3 = c_3, \ldots,\, x_n = c_n$ is a solution to the homogeneous system $\mathcal{LS}(A,\,\mathbf{0})$ formed with a nonsingular coefficient matrix. This means that the only possible solution is the trivial solution, so $c_1 = 0,\, c_2 = 0,\, c_3 = 0,\, \ldots,\, c_n = 0$. So, assuming simply that $c_{n+1} = 0$, we conclude that *all* of the c_i are zero. But this contradicts our choice of the c_i as not being the trivial solution to the system $(**)$. So $c_{n+1} \neq 0$.

We now propose and verify a solution to the original system $(*)$. Set

$$x_1 = \frac{c_1}{c_{n+1}} \qquad x_2 = \frac{c_2}{c_{n+1}} \qquad x_3 = \frac{c_3}{c_{n+1}} \qquad \ldots \qquad x_n = \frac{c_n}{c_{n+1}}$$

Notice how it was necessary that we know that $c_{n+1} \neq 0$ for this step to succeed. Now, evaluate the i-th equation of system $(*)$ with this proposed solution, and recognize in the third line that c_1 through c_{n+1} appear as if they were substituted into the left-hand side of the i-th equation of system $(**)$,

$$\begin{aligned}
&a_{i1}\frac{c_1}{c_{n+1}} + a_{i2}\frac{c_2}{c_{n+1}} + a_{i3}\frac{c_3}{c_{n+1}} + \cdots + a_{in}\frac{c_n}{c_{n+1}} \\
&= \frac{1}{c_{n+1}}\left(a_{i1}c_1 + a_{i2}c_2 + a_{i3}c_3 + \cdots + a_{in}c_n\right) \\
&= \frac{1}{c_{n+1}}\left(a_{i1}c_1 + a_{i2}c_2 + a_{i3}c_3 + \cdots + a_{in}c_n - b_ic_{n+1}\right) + b_i \\
&= \frac{1}{c_{n+1}}(0) + b_i \\
&= b_i
\end{aligned}$$

Since this equation is true for every i, we have found a solution to system $(*)$. To finish, we still need to establish that this solution is *unique*.

With one solution in hand, we will entertain the possibility of a second solution. So assume system $(*)$ has two solutions,

$$\begin{array}{lllll}
x_1 = d_1 & x_2 = d_2 & x_3 = d_3 & \ldots & x_n = d_n \\
x_1 = e_1 & x_2 = e_2 & x_3 = e_3 & \ldots & x_n = e_n
\end{array}$$

Then,

$$\begin{aligned}
&(a_{i1}(d_1 - e_1) + a_{i2}(d_2 - e_2) + a_{i3}(d_3 - e_3) + \cdots + a_{in}(d_n - e_n)) \\
&= (a_{i1}d_1 + a_{i2}d_2 + a_{i3}d_3 + \cdots + a_{in}d_n) - (a_{i1}e_1 + a_{i2}e_2 + a_{i3}e_3 + \cdots + a_{in}e_n) \\
&= b_i - b_i \\
&= 0
\end{aligned}$$

This is the i-th equation of the homogeneous system $\mathcal{LS}(A,\, \mathbf{0})$ evaluated with $x_j = d_j - e_j$, $1 \leq j \leq n$. Since A is nonsingular, we must conclude that this solution is the trivial solution, and so $0 = d_j - e_j$, $1 \leq j \leq n$. That is, $d_j = e_j$ for all j and the two solutions are identical, meaning any solution to $(*)$ is unique.

Notice that the proposed solution $(x_i = \frac{c_i}{c_{n+1}})$ appeared in this proof with no motivation whatsoever. This is just fine in a proof. A proof should *convince* you that a theorem is *true*. It is your job to *read* the proof and be convinced of every assertion. Questions like "Where did that come from?" or "How would I think of that?" have no bearing on the *validity* of the proof.

Chapter V
Vectors

Section VO
Vector Operations

C10 (Robert Beezer) Compute

$$4\begin{bmatrix}2\\-3\\4\\1\\0\end{bmatrix}+(-2)\begin{bmatrix}1\\2\\-5\\2\\4\end{bmatrix}+\begin{bmatrix}-1\\3\\0\\1\\2\end{bmatrix}$$

Solution (Robert Beezer) $\begin{bmatrix}5\\-13\\26\\1\\-6\end{bmatrix}$

C11 (Chris Black) Solve the given vector equation for x, or explain why no solution exists:

$$3\begin{bmatrix}1\\2\\-1\end{bmatrix}+4\begin{bmatrix}2\\0\\x\end{bmatrix}=\begin{bmatrix}11\\6\\17\end{bmatrix}$$

Solution (Chris Black) Performing the indicated operations (Definition CVA, Definition CVSM), we obtain the vector equations

$$\begin{bmatrix}11\\6\\17\end{bmatrix}=3\begin{bmatrix}1\\2\\-1\end{bmatrix}+4\begin{bmatrix}2\\0\\x\end{bmatrix}=\begin{bmatrix}11\\6\\-3+4x\end{bmatrix}$$

Since the entries of the vectors must be equal by Definition CVE, we have $-3+4x=17$, which leads to $x=5$.

C12 (Chris Black) Solve the given vector equation for α, or explain why no solution exists:

$$\alpha\begin{bmatrix}1\\2\\-1\end{bmatrix}+4\begin{bmatrix}3\\4\\2\end{bmatrix}=\begin{bmatrix}-1\\0\\4\end{bmatrix}$$

Solution (Chris Black) Performing the indicated operations (Definition CVA, Definition CVSM), we obtain the vector equations

$$\begin{bmatrix}\alpha\\2\alpha\\-\alpha\end{bmatrix}+\begin{bmatrix}12\\16\\8\end{bmatrix}=\begin{bmatrix}\alpha+12\\2\alpha+16\\-\alpha+8\end{bmatrix}=\begin{bmatrix}-1\\0\\4\end{bmatrix}$$

Thus, if a solution α exists, by Definition CVE then α must satisfy the three equations:

$$\begin{aligned}\alpha+12&=-1\\2\alpha+16&=0\\-\alpha+8&=4\end{aligned}$$

which leads to $\alpha=-13$, $\alpha=-8$ and $\alpha=4$. Since α cannot simultaneously have three different values, there is no solution to the original vector equation.

C13 (Chris Black) Solve the given vector equation for α, or explain why no solution exists:

$$\alpha\begin{bmatrix}3\\2\\-2\end{bmatrix}+\begin{bmatrix}6\\1\\2\end{bmatrix}=\begin{bmatrix}0\\-3\\6\end{bmatrix}$$

Solution (Chris Black) Performing the indicated operations (Definition CVA, Definition CVSM), we obtain the vector equations

$$\begin{bmatrix}3\alpha\\2\alpha\\-2\alpha\end{bmatrix}+\begin{bmatrix}6\\1\\2\end{bmatrix}=\begin{bmatrix}3\alpha+6\\2\alpha+1\\-2\alpha+2\end{bmatrix}=\begin{bmatrix}0\\-3\\6\end{bmatrix}$$

Thus, if a solution α exists, by Definition CVE then α must satisfy the three equations:

$$\begin{aligned}3\alpha+6&=0\\2\alpha+1&=-3\\-2\alpha+2&=6\end{aligned}$$

which leads to $3\alpha=-6$, $2\alpha=-4$ and $-2\alpha=4$. And thus, the solution to the given vector equation is $\alpha=-2$.

C14 (Chris Black) Find α and β that solve the vector equation.

$$\alpha\begin{bmatrix}1\\0\end{bmatrix}+\beta\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}$$

Solution (Chris Black) Performing the indicated operations (Definition CVA, Definition CVSM), we obtain the vector equations

$$\begin{bmatrix}3\\2\end{bmatrix}=\alpha\begin{bmatrix}1\\0\end{bmatrix}+\beta\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}\alpha+0\\0+\beta\end{bmatrix}=\begin{bmatrix}\alpha\\\beta\end{bmatrix}$$

Since the entries of the vectors must be equal by Definition CVE, we have $\alpha=3$ and $\beta=2$.

C15 (Chris Black) Find α and β that solve the vector equation.

$$\alpha\begin{bmatrix}2\\1\end{bmatrix}+\beta\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}5\\0\end{bmatrix}$$

Solution (Chris Black) Performing the indicated operations (Definition CVA, Definition CVSM), we obtain the vector equations

$$\begin{bmatrix}5\\0\end{bmatrix}=\alpha\begin{bmatrix}2\\1\end{bmatrix}+\beta\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}2\alpha+\beta\\\alpha+3\beta\end{bmatrix}$$

Since the entries of the vectors must be equal by Definition CVE, we obtain the system of equations

$$\begin{aligned}2\alpha+\beta&=5\\\alpha+3\beta&=0.\end{aligned}$$

which we can solve by row-reducing the augmented matrix of the system,

$$\begin{bmatrix}2&1&5\\1&3&0\end{bmatrix}\xrightarrow{\text{RREF}}\begin{bmatrix}\boxed{1}&0&3\\0&\boxed{1}&-1\end{bmatrix}$$

Thus, the only solution is $\alpha=3$, $\beta=-1$.

T05 (Chris Black) Provide reasons (mostly vector space properties) as justification for each of the seven steps of the following proof.

Theorem For any vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}\in\mathbb{C}^m$, if $\mathbf{u}+\mathbf{v}=\mathbf{u}+\mathbf{w}$, then $\mathbf{v}=\mathbf{w}$.

Proof: Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}\in\mathbb{C}^m$, and suppose $\mathbf{u}+\mathbf{v}=\mathbf{u}+\mathbf{w}$.

$$\mathbf{v}=\mathbf{0}+\mathbf{v}$$

$$\begin{aligned}
&= (-\mathbf{u} + \mathbf{u}) + \mathbf{v} \\
&= -\mathbf{u} + (\mathbf{u} + \mathbf{v}) \\
&= -\mathbf{u} + (\mathbf{u} + \mathbf{w}) \\
&= (-\mathbf{u} + \mathbf{u}) + \mathbf{w} \\
&= \mathbf{0} + \mathbf{w} \\
&= \mathbf{w}
\end{aligned}$$

Solution (Chris Black)

$$\begin{aligned}
\mathbf{v} &= \mathbf{0} + \mathbf{v} && \text{Property ZC} \\
&= (-\mathbf{u} + \mathbf{u}) + \mathbf{v} && \text{Property AIC} \\
&= -\mathbf{u} + (\mathbf{u} + \mathbf{v}) && \text{Property AAC} \\
&= -\mathbf{u} + (\mathbf{u} + \mathbf{w}) && \text{Hypothesis} \\
&= (-\mathbf{u} + \mathbf{u}) + \mathbf{w} && \text{Property AAC} \\
&= \mathbf{0} + \mathbf{w} && \text{Property AIC} \\
&= \mathbf{w} && \text{Property ZC}
\end{aligned}$$

T06 (Chris Black) Provide reasons (mostly vector space properties) as justification for each of the six steps of the following proof.

Theorem For any vector $\mathbf{u} \in \mathbb{C}^m$, $0\mathbf{u} = \mathbf{0}$.

Proof: Let $\mathbf{u} \in \mathbb{C}^m$.

$$\begin{aligned}
\mathbf{0} &= 0\mathbf{u} + (-0\mathbf{u}) \\
&= (0 + 0)\mathbf{u} + (-0\mathbf{u}) \\
&= (0\mathbf{u} + 0\mathbf{u}) + (-0\mathbf{u}) \\
&= 0\mathbf{u} + (0\mathbf{u} + (-0\mathbf{u})) \\
&= 0\mathbf{u} + \mathbf{0} \\
&= 0\mathbf{u}
\end{aligned}$$

Solution (Chris Black)

$$\begin{aligned}
\mathbf{0} &= 0\mathbf{u} + (-0\mathbf{u}) && \text{Property AIC} \\
&= (0 + 0)\mathbf{u} + (-0\mathbf{u}) && \text{Property ZCN} \\
&= (0\mathbf{u} + 0\mathbf{u}) + (-0\mathbf{u}) && \text{Property DVAC} \\
&= 0\mathbf{u} + (0\mathbf{u} + (-0\mathbf{u})) && \text{Property AAC} \\
&= 0\mathbf{u} + \mathbf{0} && \text{Property AIC} \\
&= 0\mathbf{u} && \text{Property ZC}
\end{aligned}$$

T07 (Chris Black) Provide reasons (mostly vector space properties) as justification for each of the six steps of the following proof.

Theorem For any scalar c, $c\,\mathbf{0} = \mathbf{0}$.

Proof: Let c be an arbitrary scalar.

$$\begin{aligned}
\mathbf{0} &= c\mathbf{0} + (-c\mathbf{0}) \\
&= c(\mathbf{0} + \mathbf{0}) + (-c\mathbf{0}) \\
&= (c\mathbf{0} + c\mathbf{0}) + (-c\mathbf{0})
\end{aligned}$$

$$
\begin{aligned}
&= c\mathbf{0} + (c\mathbf{0} + (-c\mathbf{0}))\\
&= c\mathbf{0} + \mathbf{0}\\
&= c\mathbf{0}
\end{aligned}
$$

Solution (Chris Black)

$$
\begin{aligned}
\mathbf{0} &= c\mathbf{0} + (-c\mathbf{0}) && \text{Property AIC}\\
&= c(\mathbf{0} + \mathbf{0}) + (-c\mathbf{0}) && \text{Property ZC}\\
&= (c\mathbf{0} + c\mathbf{0}) + (-c\mathbf{0}) && \text{Property DVAC}\\
&= c\mathbf{0} + (c\mathbf{0} + (-c\mathbf{0})) && \text{Property AAC}\\
&= c\mathbf{0} + \mathbf{0} && \text{Property AIC}\\
&= c\mathbf{0} && \text{Property ZC}
\end{aligned}
$$

T13 (Robert Beezer) Prove Property CC of Theorem VSPCV. Write your proof in the style of the proof of Property DSAC given in this section.

Solution (Robert Beezer) For all $1 \leq i \leq m$,

$$
\begin{aligned}
[\mathbf{u}+\mathbf{v}]_i &= [\mathbf{u}]_i + [\mathbf{v}]_i && \text{Definition CVA}\\
&= [\mathbf{v}]_i + [\mathbf{u}]_i && \text{Commutativity in } \mathbb{C}\\
&= [\mathbf{v}+\mathbf{u}]_i && \text{Definition CVA}
\end{aligned}
$$

With equality of each component of the vectors $\mathbf{u}+\mathbf{v}$ and $\mathbf{v}+\mathbf{u}$ being equal Definition CVE tells us the two vectors are equal.

T17 (Robert Beezer) Prove Property SMAC of Theorem VSPCV. Write your proof in the style of the proof of Property DSAC given in this section.

T18 (Robert Beezer) Prove Property DVAC of Theorem VSPCV. Write your proof in the style of the proof of Property DSAC given in this section.

Exercises T30, T31 and T32 are about making a careful definition of "vector subtraction".

T30 (Robert Beezer) Suppose $\mathbf{u}$ and $\mathbf{v}$ are two vectors in $\mathbb{C}^m$. Define a new operation, called "subtraction," as the new vector denoted $\mathbf{u}-\mathbf{v}$ and defined by

$$
[\mathbf{u}-\mathbf{v}]_i = [\mathbf{u}]_i - [\mathbf{v}]_i \qquad 1 \leq i \leq m
$$

Prove that we can express the subtraction of two vectors in terms of our two basic operations. More precisely, prove that $\mathbf{u}-\mathbf{v} = \mathbf{u} + (-1)\mathbf{v}$. So in a sense, subtraction is not something new and different, but is just a convenience. Mimic the style of similar proofs in this section.

T31 (Robert Beezer) Prove, by using counterexamples, that vector subtraction is not commutative and not associative.

T32 (Robert Beezer) Prove that vector subtraction obeys a distributive property. Specifically, prove that $\alpha(\mathbf{u}-\mathbf{v}) = \alpha\mathbf{u} - \alpha\mathbf{v}$.
Can you give two different proofs? Base one on the definition given in Exercise VO.T30 and base the other on the equivalent formulation proved in Exercise VO.T30.

Section LC
Linear Combinations

C21 (Robert Beezer) Consider each archetype that is a system of equations. For individual solutions listed (both for the original system and the corresponding homogeneous system) express the vector of constants as a linear combination of the columns of the coefficient matrix, as guaranteed by Theorem SLSLC. Verify this equality by computing the linear combination. For systems with no solutions, recognize that it is then impossible to write the vector of constants as a linear combination of the columns of the coefficient matrix. Note too, for homogeneous systems, that the solutions give rise to linear combinations that equal the zero vector.

Archetype A, Archetype B, Archetype C, Archetype D, Archetype E, Archetype F, Archetype G, Archetype H, Archetype I, Archetype J

Solution (Robert Beezer) Solutions for Archetype A and Archetype B are described carefully in Example AALC and Example ABLC.

C22 (Robert Beezer) Consider each archetype that is a system of equations. Write elements of the solution set in vector form, as guaranteed by Theorem VFSLS.

Archetype A, Archetype B, Archetype C, Archetype D, Archetype E, Archetype F, Archetype G, Archetype H, Archetype I, Archetype J

Solution (Robert Beezer) Solutions for Archetype D and Archetype I are described carefully in Example VFSAD and Example VFSAI. The technique described in these examples is probably more useful than carefully deciphering the notation of Theorem VFSLS. The solution for each archetype is contained in its description. So now you can check-off the box for that item.

C40 (Robert Beezer) Find the vector form of the solutions to the system of equations below.

$$\begin{aligned}
2x_1 - 4x_2 + 3x_3 + x_5 &= 6\\
x_1 - 2x_2 - 2x_3 + 14x_4 - 4x_5 &= 15\\
x_1 - 2x_2 + x_3 + 2x_4 + x_5 &= -1\\
-2x_1 + 4x_2 - 12x_4 + x_5 &= -7
\end{aligned}$$

Solution (Robert Beezer) Row-reduce the augmented matrix representing this system, to find

$$\begin{bmatrix}
\boxed{1} & -2 & 0 & 6 & 0 & 1\\
0 & 0 & \boxed{1} & -4 & 0 & 3\\
0 & 0 & 0 & 0 & \boxed{1} & -5\\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

The system is consistent (no leading one in column 6, Theorem RCLS). x_2 and x_4 are the free variables. Now apply Theorem VFSLS directly, or follow the three-step process of Example VFS, Example VFSAD, Example VFSAI, or Example VFSAL to obtain

$$\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{bmatrix} = \begin{bmatrix} 1\\ 0\\ 3\\ 0\\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2\\ 1\\ 0\\ 0\\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6\\ 0\\ 4\\ 1\\ 0 \end{bmatrix}$$

C41 (Robert Beezer) Find the vector form of the solutions to the system of equations below.

$$\begin{aligned}
-2x_1 - 1x_2 - 8x_3 + 8x_4 + 4x_5 - 9x_6 - 1x_7 - 1x_8 - 18x_9 &= 3\\
3x_1 - 2x_2 + 5x_3 + 2x_4 - 2x_5 - 5x_6 + 1x_7 + 2x_8 + 15x_9 &= 10\\
4x_1 - 2x_2 + 8x_3 + 2x_5 - 14x_6 - 2x_8 + 2x_9 &= 36\\
-1x_1 + 2x_2 + 1x_3 - 6x_4 + 7x_6 - 1x_7 - 3x_9 &= -8\\
3x_1 + 2x_2 + 13x_3 - 14x_4 - 1x_5 + 5x_6 - 1x_8 + 12x_9 &= 15\\
-2x_1 + 2x_2 - 2x_3 - 4x_4 + 1x_5 + 6x_6 - 2x_7 - 2x_8 - 15x_9 &= -7
\end{aligned}$$

Solution (Robert Beezer) Row-reduce the augmented matrix representing this system, to find

$$\begin{bmatrix} \boxed{1} & 0 & 3 & -2 & 0 & -1 & 0 & 0 & 3 & 6 \\ 0 & \boxed{1} & 2 & -4 & 0 & 3 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & \boxed{1} & -2 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is consistent (no leading one in column 10, Theorem RCLS). $F = \{3,\,4,\,6,\,9,\,10\}$, so the free variables are x_3, x_4, x_6 and x_9. Now apply Theorem VFSLS directly, or follow the three-step process of Example VFS, Example VFSAD, Example VFSAI, or Example VFSAL to obtain the solution set

$$S = \left\{ \begin{bmatrix} 6 \\ -1 \\ 0 \\ 0 \\ 3 \\ 0 \\ 0 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_9 \begin{bmatrix} -3 \\ -2 \\ 0 \\ 0 \\ 1 \\ 0 \\ -4 \\ -2 \\ 1 \end{bmatrix} \;\middle|\; x_3,\,x_4,\,x_6,\,x_9 \in \mathbb{C} \right\}$$

M10 (Robert Beezer) Example TLC asks if the vector

$$\mathbf{w} = \begin{bmatrix} 13 \\ 15 \\ 5 \\ -17 \\ 2 \\ 25 \end{bmatrix}$$

can be written as a linear combination of the four vectors

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 4 \\ -3 \\ 1 \\ 2 \\ 9 \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} 6 \\ 3 \\ 0 \\ -2 \\ 1 \\ 4 \end{bmatrix} \qquad \mathbf{u}_3 = \begin{bmatrix} -5 \\ 2 \\ 1 \\ 1 \\ -3 \\ 0 \end{bmatrix} \qquad \mathbf{u}_4 = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 7 \\ 1 \\ 3 \end{bmatrix}$$

Can it? Can any vector in $\mathbb{C}^6$ be written as a linear combination of the four vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$?

Solution (Robert Beezer) No, it is not possible to create $\mathbf{w}$ as a linear combination of the four vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$. By creating the desired linear combination with unknowns as scalars, Theorem SLSLC provides a system of equations that has no solution. This one computation is enough to show us that it is not possible to create all the vectors of $\mathbb{C}^6$ through linear combinations of the four vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$.

M11 (Robert Beezer) At the end of Example VFS, the vector $\mathbf{w}$ is claimed to be a solution to the linear system under discussion. Verify that $\mathbf{w}$ really is a solution. Then determine the four scalars that express $\mathbf{w}$ as a linear combination of $\mathbf{c}$, $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$.

Solution (Robert Beezer) The coefficient of $\mathbf{c}$ is 1. The coefficients of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ lie in the third, fourth and seventh entries of $\mathbf{w}$. Can you see why? (Hint: $F = \{3,\,4,\,7,\,8\}$, so the free variables are x_3, x_4 and x_7.)

Section SS
Spanning Sets

C22 (Robert Beezer) For each archetype that is a system of equations, consider the corresponding homogeneous system of equations. Write elements of the solution set to these homogeneous systems in vector form, as guaranteed by Theorem VFSLS. Then write the null space of the coefficient matrix of each system as the span of a set of vectors, as described in Theorem SSNS.

Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J

Solution (Robert Beezer) The vector form of the solutions obtained in this manner will involve precisely the vectors described in Theorem SSNS as providing the null space of the coefficient matrix of the system as a span. These vectors occur in each archetype in a description of the null space. Studying Example VFSAL may be of some help.

C23 (Robert Beezer) Archetype K and Archetype L are defined as matrices. Use Theorem SSNS directly to find a set S so that $\langle S\rangle$ is the null space of the matrix. Do not make any reference to the associated homogeneous system of equations in your solution.

Solution (Robert Beezer) Study Example NSDS to understand the correct approach to this question. The solution for each is listed in the Archetypes (Archetypes) themselves.

C40 (Robert Beezer) Suppose that $S=\left\{\begin{bmatrix}2\\-1\\3\\4\end{bmatrix},\begin{bmatrix}3\\2\\-2\\1\end{bmatrix}\right\}$. Let $W=\langle S\rangle$ and let $\mathbf{x}=\begin{bmatrix}5\\8\\-12\\-5\end{bmatrix}$. Is $\mathbf{x}\in W$? If so, provide an explicit linear combination that demonstrates this.

Solution (Robert Beezer) Rephrasing the question, we want to know if there are scalars α_1 and α_2 such that

$$\alpha_1\begin{bmatrix}2\\-1\\3\\4\end{bmatrix}+\alpha_2\begin{bmatrix}3\\2\\-2\\1\end{bmatrix}=\begin{bmatrix}5\\8\\-12\\-5\end{bmatrix}$$

Theorem SLSLC allows us to rephrase the question again as a quest for solutions to the system of four equations in two unknowns with an augmented matrix given by

$$\begin{bmatrix}2&3&5\\-1&2&8\\3&-2&-12\\4&1&-5\end{bmatrix}$$

This matrix row-reduces to

$$\begin{bmatrix}\boxed{1}&0&-2\\0&\boxed{1}&3\\0&0&0\\0&0&0\end{bmatrix}$$

From the form of this matrix, we can see that $\alpha_1=-2$ and $\alpha_2=3$ is an affirmative answer to our question. More convincingly,

$$(-2)\begin{bmatrix}2\\-1\\3\\4\end{bmatrix}+(3)\begin{bmatrix}3\\2\\-2\\1\end{bmatrix}=\begin{bmatrix}5\\8\\-12\\-5\end{bmatrix}$$

C41 (Robert Beezer) Suppose that $S=\left\{\begin{bmatrix}2\\-1\\3\\4\end{bmatrix},\begin{bmatrix}3\\2\\-2\\1\end{bmatrix}\right\}$. Let $W=\langle S\rangle$ and let $\mathbf{y}=\begin{bmatrix}5\\1\\3\\5\end{bmatrix}$. Is $\mathbf{y}\in W$? If so, provide an explicit linear combination that demonstrates this.

Solution (Robert Beezer) Rephrasing the question, we want to know if there are scalars α_1 and α_2 such that

$$\alpha_1\begin{bmatrix}2\\-1\\3\\4\end{bmatrix}+\alpha_2\begin{bmatrix}3\\2\\-2\\1\end{bmatrix}=\begin{bmatrix}5\\1\\3\\5\end{bmatrix}$$

Theorem SLSLC allows us to rephrase the question again as a quest for solutions to the system of four equations in two unknowns with an augmented matrix given by

$$\begin{bmatrix} 2 & 3 & 5 \\ -1 & 2 & 1 \\ 3 & -2 & 3 \\ 4 & 1 & 5 \end{bmatrix}$$

This matrix row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

With a leading 1 in the last column of this matrix (Theorem RCLS) we can see that the system of equations has no solution, so there are no values for α_1 and α_2 that will allow us to conclude that $\mathbf{y}$ is in W. So $\mathbf{y} \notin W$.

C42 (Robert Beezer) Suppose $R = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \\ 3 \\ -2 \end{bmatrix} \right\}$. Is $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ -8 \\ -4 \\ -3 \end{bmatrix}$ in $\langle R \rangle$?

Solution (Robert Beezer) Form a linear combination, with unknown scalars, of R that equals $\mathbf{y}$,

$$a_1 \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} + a_3 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -8 \\ -4 \\ -3 \end{bmatrix}$$

We want to know if there are values for the scalars that make the vector equation true since that is the definition of membership in $\langle R \rangle$. By Theorem SLSLC any such values will also be solutions to the linear system represented by the augmented matrix,

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ -1 & 1 & -1 & -1 \\ 3 & 2 & 0 & -8 \\ 4 & 2 & 3 & -4 \\ 0 & -1 & -2 & -3 \end{bmatrix}$$

Row-reducing the matrix yields,

$$\begin{bmatrix} \boxed{1} & 0 & 0 & -2 \\ 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we see that the system of equations is consistent (Theorem RCLS), and has a unique solution. This solution will provide a linear combination of the vectors in R that equals $\mathbf{y}$. So $\mathbf{y} \in R$.

C43 (Robert Beezer) Suppose $R = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \\ 3 \\ -2 \end{bmatrix} \right\}$. Is $\mathbf{z} = \begin{bmatrix} 1 \\ 1 \\ 5 \\ 3 \\ 1 \end{bmatrix}$ in $\langle R \rangle$?

Solution (Robert Beezer) Form a linear combination, with unknown scalars, of R that equals $\mathbf{z}$,

$$a_1 \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} + a_3 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \\ 3 \\ 1 \end{bmatrix}$$

We want to know if there are values for the scalars that make the vector equation true since that is the definition of membership in $\langle R \rangle$. By Theorem SLSLC any such values will also be solutions to the linear system represented by the augmented matrix,

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ -1 & 1 & -1 & 1 \\ 3 & 2 & 0 & 5 \\ 4 & 2 & 3 & 3 \\ 0 & -1 & -2 & 1 \end{bmatrix}$$

Row-reducing the matrix yields,

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

With a leading 1 in the last column, the system is inconsistent (Theorem RCLS), so there are no scalars a_1, a_2, a_3 that will create a linear combination of the vectors in R that equal $\mathbf{z}$. So $\mathbf{z} \notin R$.

C44 (Robert Beezer) Suppose that $S = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 1 \end{bmatrix} \right\}$. Let $W = \langle S \rangle$ and let $\mathbf{y} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$. Is $\mathbf{y} \in W$? If so, provide an explicit linear combination that demonstrates this.

Solution (Robert Beezer) Form a linear combination, with unknown scalars, of S that equals $\mathbf{y}$,

$$a_1 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + a_4 \begin{bmatrix} -6 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$$

We want to know if there are values for the scalars that make the vector equation true since that is the definition of membership in $\langle S \rangle$. By Theorem SLSLC any such values will also be solutions to the linear system represented by the augmented matrix,

$$\begin{bmatrix} -1 & 3 & 1 & -6 & -5 \\ 2 & 1 & 5 & 5 & 3 \\ 1 & 2 & 4 & 1 & 0 \end{bmatrix}$$

Row-reducing the matrix yields,

$$\begin{bmatrix} \boxed{1} & 0 & 2 & 3 & 2 \\ 0 & \boxed{1} & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we see that the system of equations is consistent (Theorem RCLS), and has a infinitely many solutions. Any solution will provide a linear combination of the vectors in R that equals $\mathbf{y}$. So $\mathbf{y} \in S$, for example,

$$(-10) \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + (3) \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + (2) \begin{bmatrix} -6 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$$

C45 (Robert Beezer) Suppose that $S = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 1 \end{bmatrix} \right\}$. Let $W = \langle S \rangle$ and let $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. Is $\mathbf{w} \in W$? If so, provide an explicit linear combination that demonstrates this.

Solution (Robert Beezer) Form a linear combination, with unknown scalars, of S that equals $\mathbf{w}$,

$$a_1 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + a_4 \begin{bmatrix} -6 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

We want to know if there are values for the scalars that make the vector equation true since that is the definition of membership in $\langle S \rangle$. By Theorem SLSLC any such values will also be solutions to the linear system represented by the augmented matrix,

$$\begin{bmatrix} -1 & 3 & 1 & -6 & 2 \\ 2 & 1 & 5 & 5 & 1 \\ 1 & 2 & 4 & 1 & 3 \end{bmatrix}$$

Row-reducing the matrix yields,

$$\begin{bmatrix} \boxed{1} & 0 & 2 & 3 & 0 \\ 0 & \boxed{1} & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

With a leading 1 in the last column, the system is inconsistent (Theorem RCLS), so there are no scalars a_1, a_2, a_3, a_4 that will create a linear combination of the vectors in S that equal $\mathbf{w}$. So $\mathbf{w} \notin \langle S \rangle$.

C50 (Robert Beezer) Let A be the matrix below.

1. Find a set S so that $\mathcal{N}(A) = \langle S \rangle$.

2. If $\mathbf{z} = \begin{bmatrix} 3 \\ -5 \\ 1 \\ 2 \end{bmatrix}$, then show directly that $\mathbf{z} \in \mathcal{N}(A)$.

3. Write $\mathbf{z}$ as a linear combination of the vectors in S.

$$A = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 1 & 2 & 1 & 3 \\ -1 & 0 & 1 & 1 \end{bmatrix}$$

Solution (Robert Beezer) (1) Theorem SSNS provides formulas for a set S with this property, but first we must row-reduce A

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 & -1 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 and x_4 would be the free variables in the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ and Theorem SSNS provides the set $S = \{\mathbf{z}_1, \mathbf{z}_2\}$ where

$$\mathbf{z}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{z}_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

(2) Simply employ the components of the vector $\mathbf{z}$ as the variables in the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. The three equations of this system evaluate as follows,

$$\begin{aligned} 2(3) + 3(-5) + 1(1) + 4(2) &= 0 \\ 1(3) + 2(-5) + 1(1) + 3(2) &= 0 \\ -1(3) + 0(-5) + 1(1) + 1(2) &= 0 \end{aligned}$$

Since each result is zero, $\mathbf{z}$ qualifies for membership in $\mathcal{N}(A)$.

(3) By Theorem SSNS we know this must be possible (that is the moral of this exercise). Find scalars α_1 and α_2 so that

$$\alpha_1\mathbf{z}_1 + \alpha_2\mathbf{z}_2 = \alpha_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 1 \\ 2 \end{bmatrix} = \mathbf{z}$$

Theorem SLSLC allows us to convert this question into a question about a system of four equations in two variables. The augmented matrix of this system row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A solution is $\alpha_1 = 1$ and $\alpha_2 = 2$. (Notice too that this solution is unique!)

C60 (Robert Beezer) For the matrix A below, find a set of vectors S so that the span of S equals the null space of A, $\langle S\rangle = \mathcal{N}(A)$.

$$A = \begin{bmatrix} 1 & 1 & 6 & -8 \\ 1 & -2 & 0 & 1 \\ -2 & 1 & -6 & 7 \end{bmatrix}$$

Solution (Robert Beezer) Theorem SSNS says that if we find the vector form of the solutions to the homogeneous system $\mathcal{LS}(A,\,\mathbf{0})$, then the fixed vectors (one per free variable) will have the desired property. Row-reduce A, viewing it as the augmented matrix of a homogeneous system with an invisible columns of zeros as the last column,

$$\begin{bmatrix} \boxed{1} & 0 & 4 & -5 \\ 0 & \boxed{1} & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Moving to the vector form of the solutions (Theorem VFSLS), with free variables x_3 and x_4, solutions to the consistent system (it is homogeneous, Theorem HSC) can be expressed as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

Then with S given by

$$S = \left\{ \begin{bmatrix} -4 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Theorem SSNS guarantees that

$$\mathcal{N}(A) = \langle S\rangle = \left\langle \left\{ \begin{bmatrix} -4 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$$

M10 (Chris Black) Consider the set of all size 2 vectors in the Cartesian plane $\mathbb{R}^2$.

1. Give a geometric description of the span of a single vector.
2. How can you tell if two vectors span the entire plane, without doing any row reduction or calculation?

Solution (Chris Black)

1. The span of a single vector $\mathbf{v}$ is the set of all linear combinations of that vector. Thus, $\langle \mathbf{v}\rangle = \{\,\alpha\mathbf{v} \mid \alpha \in \mathbb{R}\}$. This is the line through the origin and containing the (geometric) vector $\mathbf{v}$. Thus, if $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, then the span of $\mathbf{v}$ is the line through $(0, 0)$ and (v_1, v_2).
2. Two vectors will span the entire plane if they point in different directions, meaning that $\mathbf{u}$ does not lie on the line through $\mathbf{v}$ and vice-versa. That is, for vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$, $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbb{R}^2$ if $\mathbf{u}$ is not a multiple of $\mathbf{v}$.

M11 (Chris Black) Consider the set of all size 3 vectors in Cartesian 3-space $\mathbb{R}^3$.

1. Give a geometric description of the span of a single vector.
2. Describe the possibilities for the span of two vectors.
3. Describe the possibilities for the span of three vectors.

Solution (Chris Black)

1. The span of a single vector $\mathbf{v}$ is the set of all linear combinations of that vector. Thus, $\langle\mathbf{v}\rangle = \{\,\alpha\mathbf{v}\,|\,\alpha\in\mathbb{R}\}$. This is the line through the origin and containing the (geometric) vector $\mathbf{v}$. Thus, if $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, then the span of $\mathbf{v}$ is the line through $(0, 0, 0)$ and (v_1, v_2, v_3).

2. If the two vectors point in the same direction, then their span is the line through them. Recall that while two points determine a line, three points determine a plane. Two vectors will span a plane if they point in different directions, meaning that $\mathbf{u}$ does not lie on the line through $\mathbf{v}$ and vice-versa. The plane spanned by $\mathbf{u} = \begin{bmatrix} u_1 \\ u_1 \\ u_1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is determined by the origin and the points (u_1, u_2, u_3) and (v_1, v_2, v_3).

3. If all three vectors lie on the same line, then the span is that line. If one is a linear combination of the other two, but they are not all on the same line, then they will lie in a plane. Otherwise, the span of the set of three vectors will be all of 3-space.

M12 (Chris Black) Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

1. Find a vector $\mathbf{w}_1$, different from $\mathbf{u}$ and $\mathbf{v}$, so that $\langle\{\mathbf{u}, \mathbf{v}, \mathbf{w}_1\}\rangle = \langle\{\mathbf{u}, \mathbf{v}\}\rangle$.
2. Find a vector $\mathbf{w}_2$ so that $\langle\{\mathbf{u}, \mathbf{v}, \mathbf{w}_2\}\rangle \neq \langle\{\mathbf{u}, \mathbf{v}\}\rangle$.

Solution (Chris Black)

1. If we can find a vector $\mathbf{w_1}$ that is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, then $\langle\{\mathbf{u}, \mathbf{v}, \mathbf{w}_1\}\rangle$ will be the same set as $\langle\{\mathbf{u}, \mathbf{v}\}\rangle$. Thus, $\mathbf{w}_1$ can be any linear combination of $\mathbf{u}$ and $\mathbf{v}$. One such example is $\mathbf{w}_1 = 3\mathbf{u} - \mathbf{v} = \begin{bmatrix} 1 \\ 11 \\ -7 \end{bmatrix}$.

2. Now we are looking for a vector $\mathbf{w}_2$ that cannot be written as a linear combination of $\mathbf{u}$ and $\mathbf{v}$. How can we find such a vector? Any vector that matches two components but not the third of any element of $\langle\{\mathbf{u}, \mathbf{v}\}\rangle$ will not be in the span. (Why?) One such example is $\mathbf{w}_2 = \begin{bmatrix} 4 \\ -4 \\ 1 \end{bmatrix}$ (which is nearly $2\mathbf{v}$, but not quite).

M20 (Robert Beezer) In Example SCAD we began with the four columns of the coefficient matrix of Archetype D, and used these columns in a span construction. Then we methodically argued that we could remove the last column, then the third column, and create the same set by just doing a span construction with the first two columns. We claimed we could not go any further, and had removed as many vectors as possible. Provide a convincing argument for why a third vector cannot be removed.

M21 (Robert Beezer) In the spirit of Example SCAD, begin with the four columns of the coefficient matrix of Archetype C, and use these columns in a span construction to build the set S. Argue that S can be expressed as the span of just three of the columns of the coefficient matrix (saying exactly which three) and in the spirit of Exercise SS.M20 argue that no one of these three vectors can be removed and still have a span construction create S.

Solution (Robert Beezer) If the columns of the coefficient matrix from Archetype C are named $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$ then we can discover the equation

$$(-2)\mathbf{u}_1 + (-3)\mathbf{u}_2 + \mathbf{u}_3 + \mathbf{u}_4 = \mathbf{0}$$

by building a homogeneous system of equations and viewing a solution to the system as scalars in a linear combination via Theorem SLSLC. This particular vector equation can be rearranged to read

$$\mathbf{u}_4 = (2)\mathbf{u}_1 + (3)\mathbf{u}_2 + (-1)\mathbf{u}_3$$

This can be interpreted to mean that $\mathbf{u}_4$ is unnecessary in $\langle\{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\mathbf{u}_4\}\rangle$, so that

$$\langle\{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\mathbf{u}_4\}\rangle = \langle\{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3\}\rangle$$

If we try to repeat this process and find a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ that equals the zero vector, we will fail. The required homogeneous system of equations (via Theorem SLSLC) has only a trivial solution, which will not provide the kind of equation we need to remove one of the three remaining vectors.

T10 (Robert Beezer) Suppose that $\mathbf{v}_1,\,\mathbf{v}_2 \in \mathbb{C}^m$. Prove that

$$\langle\{\mathbf{v}_1,\,\mathbf{v}_2\}\rangle = \langle\{\mathbf{v}_1,\,\mathbf{v}_2,\,5\mathbf{v}_1 + 3\mathbf{v}_2\}\rangle$$

Solution (Robert Beezer) This is an equality of sets, so Definition SE applies.

First show that $X = \langle\{\mathbf{v}_1,\,\mathbf{v}_2\}\rangle \subseteq \langle\{\mathbf{v}_1,\,\mathbf{v}_2,\,5\mathbf{v}_1 + 3\mathbf{v}_2\}\rangle = Y$.
Choose $\mathbf{x} \in X$. Then $\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2$ for some scalars a_1 and a_2. Then,

$$\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + 0(5\mathbf{v}_1 + 3\mathbf{v}_2)$$

which qualifies $\mathbf{x}$ for membership in Y, as it is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $5\mathbf{v}_1 + 3\mathbf{v}_2$.

Now show the opposite inclusion, $Y = \langle\{\mathbf{v}_1,\,\mathbf{v}_2,\,5\mathbf{v}_1 + 3\mathbf{v}_2\}\rangle \subseteq \langle\{\mathbf{v}_1,\,\mathbf{v}_2\}\rangle = X$.
Choose $\mathbf{y} \in Y$. Then there are scalars a_1, a_2, a_3 such that

$$\mathbf{y} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3(5\mathbf{v}_1 + 3\mathbf{v}_2)$$

Rearranging, we obtain,

$$\begin{aligned}
\mathbf{y} &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3(5\mathbf{v}_1 + 3\mathbf{v}_2) \\
&= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + 5a_3\mathbf{v}_1 + 3a_3\mathbf{v}_2 && \text{Property DVAC} \\
&= a_1\mathbf{v}_1 + 5a_3\mathbf{v}_1 + a_2\mathbf{v}_2 + 3a_3\mathbf{v}_2 && \text{Property CC} \\
&= (a_1 + 5a_3)\mathbf{v}_1 + (a_2 + 3a_3)\mathbf{v}_2 && \text{Property DSAC}
\end{aligned}$$

This is an expression for $\mathbf{y}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, earning $\mathbf{y}$ membership in X. Since X is a subset of Y, and vice versa, we see that $X = Y$, as desired.

T20 (Robert Beezer) Suppose that S is a set of vectors from $\mathbb{C}^m$. Prove that the zero vector, $\mathbf{0}$, is an element of $\langle S\rangle$.

Solution (Robert Beezer) No matter what the elements of the set S are, we can choose the scalars in a linear combination to all be zero. Suppose that $S = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_p\}$. Then compute

$$\begin{aligned}
0\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 + \cdots + 0\mathbf{v}_p &= \mathbf{0} + \mathbf{0} + \mathbf{0} + \cdots + \mathbf{0} \\
&= \mathbf{0}
\end{aligned}$$

But what if we choose S to be the empty set? The *convention* is that the empty sum in Definition SSCV evaluates to "zero," in this case this is the zero vector.

T21 (Robert Beezer) Suppose that S is a set of vectors from $\mathbb{C}^m$ and $\mathbf{x},\,\mathbf{y} \in \langle S\rangle$. Prove that $\mathbf{x} + \mathbf{y} \in \langle S\rangle$.

T22 (Robert Beezer) Suppose that S is a set of vectors from $\mathbb{C}^m$, $\alpha \in \mathbb{C}$, and $\mathbf{x} \in \langle S\rangle$. Prove that $\alpha\mathbf{x} \in \langle S\rangle$.

Section LI
Linear Independence

Determine if the sets of vectors in Exercises C20–C25 are linearly independent or linearly dependent. When the set is linearly dependent, exhibit a nontrivial relation of linear dependence.

C20 (Robert Beezer) $\left\{\begin{bmatrix}1\\-2\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\3\end{bmatrix}, \begin{bmatrix}1\\5\\0\end{bmatrix}\right\}$

Solution (Robert Beezer) With three vectors from $\mathbb{C}^3$, we can form a square matrix by making these three vectors the columns of a matrix. We do so, and row-reduce to obtain,

$$\begin{bmatrix}\boxed{1} & 0 & 0\\ 0 & \boxed{1} & 0\\ 0 & 0 & \boxed{1}\end{bmatrix}$$

the 3×3 identity matrix. So by Theorem NME2 the original matrix is nonsingular and its columns are therefore a linearly independent set.

C21 (Robert Beezer) $\left\{\begin{bmatrix}-1\\2\\4\\2\end{bmatrix}, \begin{bmatrix}3\\3\\-1\\3\end{bmatrix}, \begin{bmatrix}7\\3\\-6\\4\end{bmatrix}\right\}$

Solution (Robert Beezer) Theorem LIVRN says we can answer this question by putting theses vectors into a matrix as columns and row-reducing. Doing this we obtain,

$$\begin{bmatrix}\boxed{1} & 0 & 0\\ 0 & \boxed{1} & 0\\ 0 & 0 & \boxed{1}\\ 0 & 0 & 0\end{bmatrix}$$

With $n = 3$ (3 vectors, 3 columns) and $r = 3$ (3 leading 1's) we have $n = r$ and the theorem says the vectors are linearly independent.

C22 (Robert Beezer) $\left\{\begin{bmatrix}-2\\-1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}3\\3\\6\end{bmatrix}, \begin{bmatrix}-5\\-4\\-6\end{bmatrix}, \begin{bmatrix}4\\4\\7\end{bmatrix}\right\}$

Solution (Robert Beezer) Five vectors from $\mathbb{C}^3$. Theorem MVSLD says the set is linearly dependent. Boom.

C23 (Robert Beezer) $\left\{\begin{bmatrix}1\\-2\\2\\5\\3\end{bmatrix}, \begin{bmatrix}3\\3\\1\\2\\-4\end{bmatrix}, \begin{bmatrix}2\\1\\2\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\2\\2\end{bmatrix}\right\}$

Solution (Robert Beezer) Theorem LIVRN suggests we analyze a matrix whose columns are the vectors of S,

$$A = \begin{bmatrix}1 & 3 & 2 & 1\\ -2 & 3 & 1 & 0\\ 2 & 1 & 2 & 1\\ 5 & 2 & -1 & 2\\ 3 & -4 & 1 & 2\end{bmatrix}$$

Row-reducing the matrix A yields,

$$\begin{bmatrix}\boxed{1} & 0 & 0 & 0\\ 0 & \boxed{1} & 0 & 0\\ 0 & 0 & \boxed{1} & 0\\ 0 & 0 & 0 & \boxed{1}\\ 0 & 0 & 0 & 0\end{bmatrix}$$

We see that $r = 4 = n$, where r is the number of nonzero rows and n is the number of columns. By Theorem LIVRN, the set S is linearly independent.

C24 (Robert Beezer) $\left\{\begin{bmatrix}1\\2\\-1\\0\\1\end{bmatrix}, \begin{bmatrix}3\\2\\-1\\2\\2\end{bmatrix}, \begin{bmatrix}4\\4\\-2\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\2\\-1\\-2\\0\end{bmatrix}\right\}$

Solution (Robert Beezer) Theorem LIVRN suggests we analyze a matrix whose columns are the vectors from the set,

$$A = \begin{bmatrix} 1 & 3 & 4 & -1 \\ 2 & 2 & 4 & 2 \\ -1 & -1 & -2 & -1 \\ 0 & 2 & 2 & -2 \\ 1 & 2 & 3 & 0 \end{bmatrix}$$

Row-reducing the matrix A yields,

$$\begin{bmatrix} \boxed{1} & 0 & 1 & 2 \\ 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that $r = 2 \neq 4 = n$, where r is the number of nonzero rows and n is the number of columns. By Theorem LIVRN, the set S is linearly dependent.

C25 (Robert Beezer) $\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 10 \\ -7 \\ 0 \\ 10 \\ 4 \end{bmatrix} \right\}$

Solution (Robert Beezer) Theorem LIVRN suggests we analyze a matrix whose columns are the vectors from the set,

$$A = \begin{bmatrix} 2 & 4 & 10 \\ 1 & -2 & -7 \\ 3 & 1 & 0 \\ -1 & 3 & 10 \\ 2 & 2 & 4 \end{bmatrix}$$

Row-reducing the matrix A yields,

$$\begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that $r = 2 \neq 3 = n$, where r is the number of nonzero rows and n is the number of columns. By Theorem LIVRN, the set S is linearly dependent.

C30 (Robert Beezer) For the matrix B below, find a set S that is linearly independent and spans the null space of B, that is, $\mathcal{N}(B) = \langle S \rangle$.

$$B = \begin{bmatrix} -3 & 1 & -2 & 7 \\ -1 & 2 & 1 & 4 \\ 1 & 1 & 2 & -1 \end{bmatrix}$$

Solution (Robert Beezer) The requested set is described by Theorem BNS. It is easiest to find by using the procedure of Example VFSAL. Begin by row-reducing the matrix, viewing it as the coefficient matrix of a homogeneous system of equations. We obtain,

$$\begin{bmatrix} \boxed{1} & 0 & 1 & -2 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now build the vector form of the solutions to this homogeneous system (Theorem VFSLS). The free variables are x_3 and x_4, corresponding to the columns without leading 1's,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

The desired set S is simply the constant vectors in this expression, and these are the vectors $\mathbf{z}_1$ and $\mathbf{z}_2$ described by Theorem BNS.

$$S = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

C31 (Robert Beezer) For the matrix A below, find a linearly independent set S so that the null space of A is spanned by S, that is, $\mathcal{N}(A) = \langle S \rangle$.

$$A = \begin{bmatrix} -1 & -2 & 2 & 1 & 5 \\ 1 & 2 & 1 & 1 & 5 \\ 3 & 6 & 1 & 2 & 7 \\ 2 & 4 & 0 & 1 & 2 \end{bmatrix}$$

Solution (Robert Beezer) Theorem BNS provides formulas for $n-r$ vectors that will meet the requirements of this question. These vectors are the same ones listed in Theorem VFSLS when we solve the homogeneous system $\mathcal{LS}(A, \mathbf{0})$, whose solution set is the null space (Definition NSM).

To apply Theorem BNS or Theorem VFSLS we first row-reduce the matrix, resulting in

$$B = \begin{bmatrix} \boxed{1} & 2 & 0 & 0 & 3 \\ 0 & 0 & \boxed{1} & 0 & 6 \\ 0 & 0 & 0 & \boxed{1} & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So we see that $n - r = 5 - 3 = 2$ and $F = \{2, 5\}$, so the vector form of a generic solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ -6 \\ 4 \\ 1 \end{bmatrix}$$

So we have

$$\mathcal{N}(A) = \left\langle \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -6 \\ 4 \\ 1 \end{bmatrix} \right\} \right\rangle$$

C32 (Robert Beezer) Find a set of column vectors, T, such that (1) the span of T is the null space of B, $\langle T \rangle = \mathcal{N}(B)$ and (2) T is a linearly independent set.

$$B = \begin{bmatrix} 2 & 1 & 1 & 1 \\ -4 & -3 & 1 & -7 \\ 1 & 1 & -1 & 3 \end{bmatrix}$$

Solution (Robert Beezer) The conclusion of Theorem BNS gives us everything this question asks for. We need the reduced row-echelon form of the matrix so we can determine the number of vectors in T, and their entries.

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ -4 & -3 & 1 & -7 \\ 1 & 1 & -1 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2 & -2 \\ 0 & \boxed{1} & -3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We can build the set T in immediately via Theorem BNS, but we will illustrate its construction in two steps. Since $F = \{3, 4\}$, we will have two vectors and can distribute strategically placed ones, and many zeros. Then we distribute the negatives of the appropriate entries of the non-pivot columns of the reduced row-echelon matrix.

$$T = \left\{ \begin{bmatrix} \\ \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ 0 \\ 1 \end{bmatrix} \right\} \qquad T = \left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

C33 (Robert Beezer) Find a set S so that S is linearly independent and $\mathcal{N}(A) = \langle S\rangle$, where $\mathcal{N}(A)$ is the null space of the matrix A below.

$$A = \begin{bmatrix} 2 & 3 & 3 & 1 & 4 \\ 1 & 1 & -1 & -1 & -3 \\ 3 & 2 & -8 & -1 & 1 \end{bmatrix}$$

Solution (Robert Beezer) A direct application of Theorem BNS will provide the desired set. We require the reduced row-echelon form of A.

$$\begin{bmatrix} 2 & 3 & 3 & 1 & 4 \\ 1 & 1 & -1 & -1 & -3 \\ 3 & 2 & -8 & -1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -6 & 0 & 3 \\ 0 & \boxed{1} & 5 & 0 & -2 \\ 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix}$$

The non-pivot columns have indices $F = \{3,\,5\}$. We build the desired set in two steps, first placing the requisite zeros and ones in locations based on F, then placing the negatives of the entries of columns 3 and 5 in the proper locations. This is all specified in Theorem BNS.

$$S = \left\{ \begin{bmatrix} \\ \\ 1 \\ \\ 0 \end{bmatrix},\, \begin{bmatrix} \\ \\ 0 \\ \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 6 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix},\, \begin{bmatrix} -3 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$$

C50 (Robert Beezer) Consider each archetype that is a system of equations and consider the solutions listed for the homogeneous version of the archetype. (If only the trivial solution is listed, then assume this is the only solution to the system.) From the solution set, determine if the columns of the coefficient matrix form a linearly independent or linearly dependent set. In the case of a linearly dependent set, use one of the sample solutions to provide a nontrivial relation of linear dependence on the set of columns of the coefficient matrix (Definition RLD). Indicate when Theorem MVSLD applies and connect this with the number of variables and equations in the system of equations.

Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J

C51 (Robert Beezer) For each archetype that is a system of equations consider the homogeneous version. Write elements of the solution set in vector form (Theorem VFSLS) and from this extract the vectors $\mathbf{z}_j$ described in Theorem BNS. These vectors are used in a span construction to describe the null space of the coefficient matrix for each archetype. What does it mean when we write a null space as $\langle\{\ \}\rangle$?

Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J

C52 (Robert Beezer) For each archetype that is a system of equations consider the homogeneous version. Sample solutions are given and a linearly independent spanning set is given for the null space of the coefficient matrix. Write each of the sample solutions individually as a linear combination of the vectors in the spanning set for the null space of the coefficient matrix.

Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J

C60 (Robert Beezer) For the matrix A below, find a set of vectors S so that (1) S is linearly independent, and (2) the span of S equals the null space of A, $\langle S\rangle = \mathcal{N}(A)$. (See Exercise SS.C60.)

$$A = \begin{bmatrix} 1 & 1 & 6 & -8 \\ 1 & -2 & 0 & 1 \\ -2 & 1 & -6 & 7 \end{bmatrix}$$

Solution (Robert Beezer) Theorem BNS says that if we find the vector form of the solutions to the homogeneous system $\mathcal{LS}(A,\,\mathbf{0})$, then the fixed vectors (one per free variable) will have the desired properties.

Row-reduce A, viewing it as the augmented matrix of a homogeneous system with an invisible columns of zeros as the last column,

$$\begin{bmatrix} \boxed{1} & 0 & 4 & -5 \\ 0 & \boxed{1} & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Moving to the vector form of the solutions (Theorem VFSLS), with free variables x_3 and x_4, solutions to the consistent system (it is homogeneous, Theorem HSC) can be expressed as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

Then with S given by

$$S = \left\{ \begin{bmatrix} -4 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Theorem BNS guarantees the set has the desired properties.

M20 (Robert Beezer) Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a set of three vectors from $\mathbb{C}^{873}$. Prove that the set

$$T = \{2\mathbf{v}_1 + 3\mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_1 - \mathbf{v}_2 - 2\mathbf{v}_3, 2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3\}$$

is linearly dependent.

Solution (Robert Beezer) By Definition LICV, we can complete this problem by finding scalars, α_1, α_2, α_3, not all zero, such that

$$\alpha_1 (2\mathbf{v}_1 + 3\mathbf{v}_2 + \mathbf{v}_3) + \alpha_2 (\mathbf{v}_1 - \mathbf{v}_2 - 2\mathbf{v}_3) + \alpha_3 (2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3) = \mathbf{0}$$

Using various properties in Theorem VSPCV, we can rearrange this vector equation to

$$(2\alpha_1 + \alpha_2 + 2\alpha_3) \mathbf{v}_1 + (3\alpha_1 - \alpha_2 + \alpha_3) \mathbf{v}_2 + (\alpha_1 - 2\alpha_2 - \alpha_3) \mathbf{v}_3 = \mathbf{0}$$

We can certainly make this vector equation true if we can determine values for the α's such that

$$\begin{aligned} 2\alpha_1 + \alpha_2 + 2\alpha_3 &= 0 \\ 3\alpha_1 - \alpha_2 + \alpha_3 &= 0 \\ \alpha_1 - 2\alpha_2 - \alpha_3 &= 0 \end{aligned}$$

Aah, a homogeneous system of equations. And it has infinitely many non-zero solutions. By the now familiar techniques, one such solution is $\alpha_1 = 3$, $\alpha_2 = 4$, $\alpha_3 = -5$, which you can check in the original relation of linear dependence on T above.

Note that simply writing down the three scalars, and demonstrating that they provide a nontrivial relation of linear dependence on T, could be considered an ironclad solution. But it wouldn't have been very informative for you if we had only done just that here. Compare this solution very carefully with Solution LI.M21.

M21 (Robert Beezer) Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of three vectors from $\mathbb{C}^{873}$. Prove that the set

$$T = \{2\mathbf{v}_1 + 3\mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_1 - \mathbf{v}_2 + 2\mathbf{v}_3, 2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3\}$$

is linearly independent.

Solution (Robert Beezer) By Definition LICV we can complete this problem by proving that if we assume that

$$\alpha_1 (2\mathbf{v}_1 + 3\mathbf{v}_2 + \mathbf{v}_3) + \alpha_2 (\mathbf{v}_1 - \mathbf{v}_2 + 2\mathbf{v}_3) + \alpha_3 (2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3) = \mathbf{0}$$

then we *must* conclude that $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Using various properties in Theorem VSPCV, we can rearrange this vector equation to

$$(2\alpha_1 + \alpha_2 + 2\alpha_3) \mathbf{v}_1 + (3\alpha_1 - \alpha_2 + \alpha_3) \mathbf{v}_2 + (\alpha_1 + 2\alpha_2 - \alpha_3) \mathbf{v}_3 = \mathbf{0}$$

Because the set $S = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3\}$ was assumed to be linearly independent, by Definition LICV we *must* conclude that

$$\begin{aligned} 2\alpha_1 + \alpha_2 + 2\alpha_3 &= 0\\ 3\alpha_1 - \alpha_2 + \alpha_3 &= 0\\ \alpha_1 + 2\alpha_2 - \alpha_3 &= 0 \end{aligned}$$

Aah, a homogeneous system of equations. And it has a unique solution, the trivial solution. So, $\alpha_1 = \alpha_2 = \alpha_3 = 0$, as desired. It is an inescapable conclusion from our assumption of a relation of linear dependence above. Done.

Compare this solution very carefully with Solution LI.M20, noting especially how this problem required (and used) the hypothesis that the original set be linearly independent, and how this solution feels more like a proof, while the previous problem could be solved with a fairly simple demonstration of any nontrivial relation of linear dependence.

M50 (Robert Beezer) Consider the set of vectors from $\mathbb{C}^3$, W, given below. Find a set T that contains three vectors from W and such that $W = \langle T\rangle$.

$$W = \langle\{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\mathbf{v}_4,\,\mathbf{v}_5\}\rangle = \left\langle\left\{\begin{bmatrix}2\\1\\1\end{bmatrix},\,\begin{bmatrix}-1\\-1\\1\end{bmatrix},\,\begin{bmatrix}1\\2\\3\end{bmatrix},\,\begin{bmatrix}3\\1\\3\end{bmatrix},\,\begin{bmatrix}0\\1\\-3\end{bmatrix}\right\}\right\rangle$$

Solution (Robert Beezer) We want to first find some relations of linear dependence on $\{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\mathbf{v}_4,\,\mathbf{v}_5\}$ that will allow us to "kick out" some vectors, in the spirit of Example SCAD. To find relations of linear dependence, we formulate a matrix A whose columns are $\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\mathbf{v}_4,\,\mathbf{v}_5$. Then we consider the homogeneous system of equations $\mathcal{LS}(A,\,\mathbf{0})$ by row-reducing its coefficient matrix (remember that if we formulated the augmented matrix we would just add a column of zeros). After row-reducing, we obtain

$$\begin{bmatrix}\boxed{1} & 0 & 0 & 2 & -1\\ 0 & \boxed{1} & 0 & 1 & -2\\ 0 & 0 & \boxed{1} & 0 & 0\end{bmatrix}$$

From this we find that solutions can be obtained employing the free variables x_4 and x_5. With appropriate choices we will be able to conclude that vectors $\mathbf{v}_4$ and $\mathbf{v}_5$ are unnecessary for creating W via a span. By Theorem SLSLC the choice of free variables below lead to solutions and linear combinations, which are then rearranged.

$$\begin{aligned} x_4 = 1, x_5 = 0 &\quad\Rightarrow\quad (-2)\mathbf{v}_1 + (-1)\mathbf{v}_2 + (0)\mathbf{v}_3 + (1)\mathbf{v}_4 + (0)\mathbf{v}_5 = \mathbf{0} &\quad\Rightarrow\quad \mathbf{v}_4 = 2\mathbf{v}_1 + \mathbf{v}_2\\ x_4 = 0, x_5 = 1 &\quad\Rightarrow\quad (1)\mathbf{v}_1 + (2)\mathbf{v}_2 + (0)\mathbf{v}_3 + (0)\mathbf{v}_4 + (1)\mathbf{v}_5 = \mathbf{0} &\quad\Rightarrow\quad \mathbf{v}_5 = -\mathbf{v}_1 - 2\mathbf{v}_2 \end{aligned}$$

Since $\mathbf{v}_4$ and $\mathbf{v}_5$ can be expressed as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$ we can say that $\mathbf{v}_4$ and $\mathbf{v}_5$ are not needed for the linear combinations used to build W (a claim that we could establish carefully with a pair of set equality arguments). Thus

$$W = \langle\{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3\}\rangle = \left\langle\left\{\begin{bmatrix}2\\1\\1\end{bmatrix},\,\begin{bmatrix}-1\\-1\\1\end{bmatrix},\,\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}\right\rangle$$

That the $\{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3\}$ is linearly independent set can be established quickly with Theorem LIVRN.

There are other answers to this question, but notice that any nontrivial linear combination of $\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\mathbf{v}_4,\,\mathbf{v}_5$ will have a zero coefficient on $\mathbf{v}_3$, so this vector can never be eliminated from the set used to build the span.

M51 (Manley Perkel) Consider the subspace $W = \langle\{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\mathbf{v}_4\}\rangle$. Find a set S so that (1) S is a subset of W, (2) S is linearly independent, and (3) $W = \langle S\rangle$. Write each vector not included in S as a linear combination of the vectors that are in S.

$$\mathbf{v}_1 = \begin{bmatrix}1\\-1\\2\end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix}4\\-4\\8\end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix}-3\\2\\-7\end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix}2\\1\\7\end{bmatrix}$$

Solution (Robert Beezer) This problem can be solved using the approach in Solution LI.M50. We will provide a solution here that is more ad-hoc, but note that we will have a more straight-forward procedure given by the upcoming Theorem BS.

$\mathbf{v}_1$ is a non-zero vector, so in a set all by itself we have a linearly independent set. As $\mathbf{v}_2$ is a scalar multiple of $\mathbf{v}_1$, the equation $-4\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$ is a relation of linear dependence on $\{\mathbf{v}_1,\,\mathbf{v}_2\}$, so we will pass on $\mathbf{v}_2$. No such relation of linear dependence exists on $\{\mathbf{v}_1,\,\mathbf{v}_3\}$, though on $\{\mathbf{v}_1,\,\mathbf{v}_3,\,\mathbf{v}_4\}$ we have the relation of linear dependence $7\mathbf{v}_1 + 3\mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$. So take $S = \{\mathbf{v}_1,\,\mathbf{v}_3\}$, which is linearly independent.

Then

$$\mathbf{v}_2 = 4\mathbf{v_1} + 0\mathbf{v_3} \qquad\qquad \mathbf{v}_4 = -7\mathbf{v_1} - 3\mathbf{v_3}$$

The two equations above are enough to justify the set equality

$$W = \langle\{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\mathbf{v}_4\}\rangle = \langle\{\mathbf{v}_1,\,\mathbf{v}_3\}\rangle = \langle S\rangle$$

There are other solutions (for example, swap the roles of $\mathbf{v}_1$ and $\mathbf{v}_2$, but by upcoming theorems we can confidently claim that any solution will be a set S with exactly two vectors.

T10 (Martin Jackson) Prove that if a set of vectors contains the zero vector, then the set is linearly dependent. (Ed. "The zero vector is death to linearly independent sets.")

T12 (Robert Beezer) Suppose that S is a linearly independent set of vectors, and T is a subset of S, $T \subseteq S$ (Definition SSET). Prove that T is linearly independent.

T13 (Robert Beezer) Suppose that T is a linearly dependent set of vectors, and T is a subset of S, $T \subseteq S$ (Definition SSET). Prove that S is linearly dependent.

T15 (Robert Beezer) Suppose that $\{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_n\}$ is a set of vectors. Prove that

$$\{\mathbf{v}_1 - \mathbf{v}_2,\,\mathbf{v}_2 - \mathbf{v}_3,\,\mathbf{v}_3 - \mathbf{v}_4,\,\ldots,\,\mathbf{v}_n - \mathbf{v}_1\}$$

is a linearly dependent set.

Solution (Robert Beezer) Consider the following linear combination

$$\begin{aligned} 1\left(\mathbf{v}_1 - \mathbf{v}_2\right) &+ 1\left(\mathbf{v}_2 - \mathbf{v}_3\right) + 1\left(\mathbf{v}_3 - \mathbf{v}_4\right) + \cdots + 1\left(\mathbf{v}_n - \mathbf{v}_1\right) \\ &= \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_2 - \mathbf{v}_3 + \mathbf{v}_3 - \mathbf{v}_4 + \cdots + \mathbf{v}_n - \mathbf{v}_1 \\ &= \mathbf{v}_1 + \mathbf{0} + \mathbf{0} + \cdots + \mathbf{0} - \mathbf{v}_1 \\ &= \mathbf{0} \end{aligned}$$

This is a nontrivial relation of linear dependence (Definition RLDCV), so by Definition LICV the set is linearly dependent.

T20 (Robert Beezer) Suppose that $\{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\mathbf{v}_4\}$ is a linearly independent set in $\mathbb{C}^{35}$. Prove that

$$\{\mathbf{v}_1,\,\mathbf{v}_1 + \mathbf{v}_2,\,\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3,\,\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4\}$$

is a linearly independent set.

Solution (Robert Beezer) Our hypothesis and our conclusion use the term linear independence, so it will get a workout. To establish linear independence, we begin with the definition (Definition LICV) and write a relation of linear dependence (Definition RLDCV),

$$\alpha_1\left(\mathbf{v}_1\right) + \alpha_2\left(\mathbf{v}_1 + \mathbf{v}_2\right) + \alpha_3\left(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\right) + \alpha_4\left(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4\right) = \mathbf{0}$$

Using the distributive and commutative properties of vector addition and scalar multiplication (Theorem VSPCV) this equation can be rearranged as

$$\left(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\right)\mathbf{v}_1 + \left(\alpha_2 + \alpha_3 + \alpha_4\right)\mathbf{v}_2 + \left(\alpha_3 + \alpha_4\right)\mathbf{v}_3 + \left(\alpha_4\right)\mathbf{v}_4 = \mathbf{0}$$

However, this is a relation of linear dependence (Definition RLDCV) on a linearly independent set, $\{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\mathbf{v}_4\}$ (this was our lone hypothesis). By the definition of linear independence (Definition LICV) the scalars must all be zero. This is the homogeneous system of equations,

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$$

$$\begin{aligned} \alpha_2 + \alpha_3 + \alpha_4 &= 0 \\ \alpha_3 + \alpha_4 &= 0 \\ \alpha_4 &= 0 \end{aligned}$$

Row-reducing the coefficient matrix of this system (or backsolving) gives the conclusion

$$\alpha_1 = 0 \qquad \alpha_2 = 0 \qquad \alpha_3 = 0 \qquad \alpha_4 = 0$$

This means, by Definition LICV, that the original set

$$\{\mathbf{v}_1,\, \mathbf{v}_1 + \mathbf{v}_2,\, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3,\, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4\}$$

is linearly independent.

T50 (Robert Beezer) Suppose that A is an $m \times n$ matrix with linearly independent columns and the linear system $\mathcal{LS}(A,\, \mathbf{b})$ is consistent. Show that this system has a unique solution. (Notice that we are not requiring A to be square.)

Solution (Robert Beezer) Let $A = [\mathbf{A}_1|\mathbf{A}_2|\mathbf{A}_3|\dots|\mathbf{A}_n]$. $\mathcal{LS}(A,\, \mathbf{b})$ is consistent, so we know the system has at least one solution (Definition CS). We would like to show that there are no more than one solution to the system. Employing Proof Technique U, suppose that $\mathbf{x}$ and $\mathbf{y}$ are two solution vectors for $\mathcal{LS}(A,\, \mathbf{b})$. By Theorem SLSLC we know we can write,

$$\begin{aligned} \mathbf{b} &= [\mathbf{x}]_1 A_1 + [\mathbf{x}]_2 A_2 + [\mathbf{x}]_3 A_3 + \cdots + [\mathbf{x}]_n A_n \\ \mathbf{b} &= [\mathbf{y}]_1 A_1 + [\mathbf{y}]_2 A_2 + [\mathbf{y}]_3 A_3 + \cdots + [\mathbf{y}]_n A_n \end{aligned}$$

Then

$$\begin{aligned} \mathbf{0} &= \mathbf{b} - \mathbf{b} \\ &= ([\mathbf{x}]_1 A_1 + [\mathbf{x}]_2 A_2 + \cdots + [\mathbf{x}]_n A_n) - ([\mathbf{y}]_1 A_1 + [\mathbf{y}]_2 A_2 + \cdots + [\mathbf{y}]_n A_n) \\ &= ([\mathbf{x}]_1 - [\mathbf{y}]_1) A_1 + ([\mathbf{x}]_2 - [\mathbf{y}]_2) A_2 + \cdots + ([\mathbf{x}]_n - [\mathbf{y}]_n) A_n \end{aligned}$$

This is a relation of linear dependence (Definition RLDCV) on a linearly independent set (the columns of A). So the scalars *must* all be zero,

$$[\mathbf{x}]_1 - [\mathbf{y}]_1 = 0 \qquad [\mathbf{x}]_2 - [\mathbf{y}]_2 = 0 \qquad \dots \qquad [\mathbf{x}]_n - [\mathbf{y}]_n = 0$$

Rearranging these equations yields the statement that $[\mathbf{x}]_i = [\mathbf{y}]_i$, for $1 \leq i \leq n$. However, this is exactly how we define vector equality (Definition CVE), so $\mathbf{x} = \mathbf{y}$ and the system has only one solution.

Section LDS
Linear Dependence and Spans

C20 (Robert Beezer) Let T be the set of columns of the matrix B below. Define $W = \langle T \rangle$. Find a set R so that (1) R has 3 vectors, (2) R is a subset of T, and (3) $W = \langle R \rangle$.

$$B = \begin{bmatrix} -3 & 1 & -2 & 7 \\ -1 & 2 & 1 & 4 \\ 1 & 1 & 2 & -1 \end{bmatrix}$$

Solution (Robert Beezer) Let $T = \{\mathbf{w}_1,\, \mathbf{w}_2,\, \mathbf{w}_3,\, \mathbf{w}_4\}$. The vector $\begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ is a solution to the homogeneous system with the matrix B as the coefficient matrix (check this!). By Theorem SLSLC it provides the scalars for a linear combination of the columns of B (the vectors in T) that equals the zero vector, a relation of linear dependence on T,

$$2\mathbf{w}_1 + (-1)\mathbf{w}_2 + (1)\mathbf{w}_4 = \mathbf{0}$$

We can rearrange this equation by solving for $\mathbf{w}_4$,

$$\mathbf{w}_4 = (-2)\mathbf{w}_1 + \mathbf{w}_2$$

This equation tells us that the vector $\mathbf{w}_4$ is superfluous in the span construction that creates W. So $W = \langle\{\mathbf{w}_1,\, \mathbf{w}_2,\, \mathbf{w}_3\}\rangle$. The requested set is $R = \{\mathbf{w}_1,\, \mathbf{w}_2,\, \mathbf{w}_3\}$.

C40 (Robert Beezer) Verify that the set $R' = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_4\}$ at the end of Example RSC5 is linearly independent.

C50 (Robert Beezer) Consider the set of vectors from $\mathbb{C}^3$, W, given below. Find a linearly independent set T that contains three vectors from W and such that $\langle W\rangle = \langle T\rangle$.

$$W = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\mathbf{v}_4,\,\mathbf{v}_5\} = \left\{\begin{bmatrix}2\\1\\1\end{bmatrix},\,\begin{bmatrix}-1\\-1\\1\end{bmatrix},\,\begin{bmatrix}1\\2\\3\end{bmatrix},\,\begin{bmatrix}3\\1\\3\end{bmatrix},\,\begin{bmatrix}0\\1\\-3\end{bmatrix}\right\}$$

Solution (Robert Beezer) To apply Theorem BS, we formulate a matrix A whose columns are $\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\mathbf{v}_4,\,\mathbf{v}_5$. Then we row-reduce A. After row-reducing, we obtain

$$\begin{bmatrix}\boxed{1} & 0 & 0 & 2 & -1\\ 0 & \boxed{1} & 0 & 1 & -2\\ 0 & 0 & \boxed{1} & 0 & 0\end{bmatrix}$$

From this we see that the pivot columns are $D = \{1,\,2,\,3\}$. Thus

$$T = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3\} = \left\{\begin{bmatrix}2\\1\\1\end{bmatrix},\,\begin{bmatrix}-1\\-1\\1\end{bmatrix},\,\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$$

is a linearly independent set and $\langle T\rangle = W$. Compare this problem with Exercise LI.M50.

C51 (Robert Beezer) Given the set S below, find a linearly independent set T so that $\langle T\rangle = \langle S\rangle$.

$$S = \left\{\begin{bmatrix}2\\-1\\2\end{bmatrix},\,\begin{bmatrix}3\\0\\1\end{bmatrix},\,\begin{bmatrix}1\\1\\-1\end{bmatrix},\,\begin{bmatrix}5\\-1\\3\end{bmatrix}\right\}$$

Solution (Robert Beezer) Theorem BS says we can make a matrix with these four vectors as columns, row-reduce, and just keep the columns with indices in the set D. Here we go, forming the relevant matrix and row-reducing,

$$\begin{bmatrix}2 & 3 & 1 & 5\\ -1 & 0 & 1 & -1\\ 2 & 1 & -1 & 3\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & -1 & 1\\ 0 & \boxed{1} & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Analyzing the row-reduced version of this matrix, we see that the first two columns are pivot columns, so $D = \{1, 2\}$. Theorem BS says we need only "keep" the first two columns to create a set with the requisite properties,

$$T = \left\{\begin{bmatrix}2\\-1\\2\end{bmatrix},\,\begin{bmatrix}3\\0\\1\end{bmatrix}\right\}$$

C52 (Robert Beezer) Let W be the span of the set of vectors S below, $W = \langle S\rangle$. Find a set T so that 1) the span of T is W, $\langle T\rangle = W$, (2) T is a linearly independent set, and (3) T is a subset of S.

$$S = \left\{\begin{bmatrix}1\\2\\-1\end{bmatrix},\,\begin{bmatrix}2\\-3\\1\end{bmatrix},\,\begin{bmatrix}4\\1\\-1\end{bmatrix},\,\begin{bmatrix}3\\1\\1\end{bmatrix},\,\begin{bmatrix}3\\-1\\0\end{bmatrix}\right\}$$

Solution (Robert Beezer) This is a straight setup for the conclusion of Theorem BS. The hypotheses of this theorem tell us to pack the vectors of W into the columns of a matrix and row-reduce,

$$\begin{bmatrix}1 & 2 & 4 & 3 & 3\\ 2 & -3 & 1 & 1 & -1\\ -1 & 1 & -1 & 1 & 0\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & 2 & 0 & 1\\ 0 & \boxed{1} & 1 & 0 & 1\\ 0 & 0 & 0 & \boxed{1} & 0\end{bmatrix}$$

Pivot columns have indices $D = \{1,\,2,\,4\}$. Theorem BS tells us to form T with columns 1, 2 and 4 of S,

$$S = \left\{\begin{bmatrix}1\\2\\-1\end{bmatrix},\,\begin{bmatrix}2\\-3\\1\end{bmatrix},\,\begin{bmatrix}3\\1\\1\end{bmatrix}\right\}$$

C55 (Robert Beezer) Let T be the set of vectors $T = \left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}3\\0\\1\end{bmatrix}, \begin{bmatrix}4\\2\\3\end{bmatrix}, \begin{bmatrix}3\\0\\6\end{bmatrix}\right\}$. Find two different subsets of T, named R and S, so that R and S each contain three vectors, and so that $\langle R\rangle = \langle T\rangle$ and $\langle S\rangle = \langle T\rangle$. Prove that both R and S are linearly independent.

Solution (Robert Beezer) Let A be the matrix whose columns are the vectors in T. Then row-reduce A,

$$A \xrightarrow{\text{RREF}} B = \begin{bmatrix} \boxed{1} & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix}$$

From Theorem BS we can form R by choosing the columns of A that correspond to the pivot columns of B. Theorem BS also guarantees that R will be linearly independent.

$$R = \left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}3\\0\\1\end{bmatrix}, \begin{bmatrix}4\\2\\3\end{bmatrix}\right\}$$

That was easy. To find S will require a bit more work. From B we can obtain a solution to $\mathcal{LS}(A, \mathbf{0})$, which by Theorem SLSLC will provide a nontrivial relation of linear dependence on the columns of A, which are the vectors in T. To wit, choose the free variable x_4 to be 1, then $x_1 = -2$, $x_2 = 1$, $x_3 = -1$, and so

$$(-2)\begin{bmatrix}1\\-1\\2\end{bmatrix} + (1)\begin{bmatrix}3\\0\\1\end{bmatrix} + (-1)\begin{bmatrix}4\\2\\3\end{bmatrix} + (1)\begin{bmatrix}3\\0\\6\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

this equation can be rewritten with the second vector staying put, and the other three moving to the other side of the equality,

$$\begin{bmatrix}3\\0\\1\end{bmatrix} = (2)\begin{bmatrix}1\\-1\\2\end{bmatrix} + (1)\begin{bmatrix}4\\2\\3\end{bmatrix} + (-1)\begin{bmatrix}3\\0\\6\end{bmatrix}$$

We could have chosen other vectors to stay put, but may have then needed to divide by a nonzero scalar. This equation is enough to conclude that the second vector in T is "surplus" and can be replaced (see the careful argument in Example RSC5). So set

$$S = \left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}4\\2\\3\end{bmatrix}, \begin{bmatrix}3\\0\\6\end{bmatrix}\right\}$$

and then $\langle S\rangle = \langle T\rangle$. T is also a linearly independent set, which we can show directly. Make a matrix C whose columns are the vectors in S. Row-reduce B and you will obtain the identity matrix I_3. By Theorem LIVRN, the set S is linearly independent.

C70 (Robert Beezer) Reprise Example RES by creating a new version of the vector $\mathbf{y}$. In other words, form a new, different linear combination of the vectors in R to create a new vector $\mathbf{y}$ (but do not simplify the problem too much by choosing any of the five new scalars to be zero). Then express this new $\mathbf{y}$ as a combination of the vectors in P.

M10 (Robert Beezer) At the conclusion of Example RSC4 two alternative solutions, sets T' and T^*, are proposed. Verify these claims by proving that $\langle T\rangle = \langle T'\rangle$ and $\langle T\rangle = \langle T^*\rangle$.

T40 (Robert Beezer) Suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are any two vectors from $\mathbb{C}^m$. Prove the following set equality.

$$\langle\{\mathbf{v}_1, \mathbf{v}_2\}\rangle = \langle\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\}\rangle$$

Solution (Robert Beezer) This is an equality of sets, so Definition SE applies.

The "easy" half first. Show that $X = \langle\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\}\rangle \subseteq \langle\{\mathbf{v}_1, \mathbf{v}_2\}\rangle = Y$.
Choose $\mathbf{x} \in X$. Then $\mathbf{x} = a_1(\mathbf{v}_1 + \mathbf{v}_2) + a_2(\mathbf{v}_1 - \mathbf{v}_2)$ for some scalars a_1 and a_2. Then,

$$\begin{aligned}\mathbf{x} &= a_1(\mathbf{v}_1 + \mathbf{v}_2) + a_2(\mathbf{v}_1 - \mathbf{v}_2)\\ &= a_1\mathbf{v}_1 + a_1\mathbf{v}_2 + a_2\mathbf{v}_1 + (-a_2)\mathbf{v}_2\\ &= (a_1 + a_2)\mathbf{v}_1 + (a_1 - a_2)\mathbf{v}_2\end{aligned}$$

which qualifies $\mathbf{x}$ for membership in Y, as it is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$.

Now show the opposite inclusion, $Y = \langle\{\mathbf{v}_1,\, \mathbf{v}_2\}\rangle \subseteq \langle\{\mathbf{v}_1 + \mathbf{v}_2,\, \mathbf{v}_1 - \mathbf{v}_2\}\rangle = X$.
Choose $\mathbf{y} \in Y$. Then there are scalars b_1, b_2 such that $\mathbf{y} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2$. Rearranging, we obtain,

$$\begin{aligned}
\mathbf{y} &= b_1\mathbf{v}_1 + b_2\mathbf{v}_2\\
&= \frac{b_1}{2}\left[(\mathbf{v}_1 + \mathbf{v}_2) + (\mathbf{v}_1 - \mathbf{v}_2)\right] + \frac{b_2}{2}\left[(\mathbf{v}_1 + \mathbf{v}_2) - (\mathbf{v}_1 - \mathbf{v}_2)\right]\\
&= \frac{b_1 + b_2}{2}(\mathbf{v}_1 + \mathbf{v}_2) + \frac{b_1 - b_2}{2}(\mathbf{v}_1 - \mathbf{v}_2)
\end{aligned}$$

This is an expression for $\mathbf{y}$ as a linear combination of $\mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_1 - \mathbf{v}_2$, earning $\mathbf{y}$ membership in X. Since X is a subset of Y, and vice versa, we see that $X = Y$, as desired.

Section O
Orthogonality

C20 (Robert Beezer) Complete Example AOS by verifying that the four remaining inner products are zero.

C21 (Robert Beezer) Verify that the set T created in Example GSTV by the Gram-Schmidt Procedure is an orthogonal set.

M60 (Manley Perkel) Suppose that $\{\mathbf{u},\, \mathbf{v},\, \mathbf{w}\} \subseteq \mathbb{C}^n$ is an orthonormal set. Prove that $\mathbf{u} + \mathbf{v}$ is not orthogonal to $\mathbf{v} + \mathbf{w}$.

T10 (Robert Beezer) Prove part 2 of the conclusion of Theorem IPVA.

T11 (Robert Beezer) Prove part 2 of the conclusion of Theorem IPSM.

T20 (Robert Beezer) Suppose that $\mathbf{u},\, \mathbf{v},\, \mathbf{w} \in \mathbb{C}^n$, $\alpha,\, \beta \in \mathbb{C}$ and $\mathbf{u}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$. Prove that $\mathbf{u}$ is orthogonal to $\alpha\mathbf{v} + \beta\mathbf{w}$.

Solution (Robert Beezer) Vectors are orthogonal if their inner product is zero (Definition OV), so we compute,

$$\begin{aligned}
\langle\mathbf{u},\, \alpha\mathbf{v} + \beta\mathbf{w}\rangle &= \langle\mathbf{u},\, \alpha\mathbf{v}\rangle + \langle\mathbf{u},\, \beta\mathbf{w}\rangle && \text{Theorem IPVA}\\
&= \alpha\langle\mathbf{u},\, \mathbf{v}\rangle + \beta\langle\mathbf{u},\, \mathbf{w}\rangle && \text{Theorem IPSM}\\
&= \alpha(0) + \beta(0) && \text{Definition OV}\\
&= 0
\end{aligned}$$

So by Definition OV, $\mathbf{u}$ and $\alpha\mathbf{v} + \beta\mathbf{w}$ are an orthogonal pair of vectors.

T30 (Steve Canfield) Suppose that the set S in the hypothesis of Theorem GSP is not just linearly independent, but is also orthogonal. Prove that the set T created by the Gram-Schmidt procedure is equal to S. (Note that we are getting a stronger conclusion than $\langle T\rangle = \langle S\rangle$ — the conclusion is that $T = S$.) In other words, it is pointless to apply the Gram-Schmidt procedure to a set that is already orthogonal.

Chapter M
Matrices

Section MO
Matrix Operations

C10 (Chris Black) Let $A = \begin{bmatrix} 1 & 4 & -3 \\ 6 & 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 & 1 \\ -2 & -6 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 4 \\ 4 & 0 \\ -2 & 2 \end{bmatrix}$. Let $\alpha = 4$ and $\beta = 1/2$. Perform the following calculations: (1) $A+B$, (2) $A+C$, (3) B^t+C, (4) $A+B^t$, (5) βC, (6) $4A-3B$, (7) $A^t + \alpha C$, (8) $A + B - C^t$, (9) $4A + 2B - 5C^t$

Solution (Chris Black)

1. $A + B = \begin{bmatrix} 4 & 6 & -2 \\ 4 & -3 & 5 \end{bmatrix}$
2. $A + C$ is undefined; A and C are not the same size.
3. $B^t + C = \begin{bmatrix} 5 & 2 \\ 6 & -6 \\ -1 & 7 \end{bmatrix}$
4. $A + B^t$ is undefined; A and B^t are not the same size.
5. $\beta C = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$
6. $4A - 3B = \begin{bmatrix} -5 & 10 & -15 \\ 30 & 30 & -15 \end{bmatrix}$
7. $A^t + \alpha C = \begin{bmatrix} 9 & 22 \\ 20 & 3 \\ -11 & 8 \end{bmatrix}$
8. $A + B - C^t = \begin{bmatrix} 2 & 2 & 0 \\ 0 & -3 & 3 \end{bmatrix}$
9. $4A + 2B - 5C^t = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

C11 (Chris Black) Solve the given vector equation for x, or explain why no solution exists:

$$2\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & x \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 5 & -2 \end{bmatrix}$$

Solution (Chris Black) The given equation

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 5 & -2 \end{bmatrix} = 2\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & x \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 5 & 4 - 3x \end{bmatrix}$$

is valid only if $4 - 3x = -2$. Thus, the only solution is $x = 2$.

C12 (Chris Black) Solve the given matrix equation for α, or explain why no solution exists:

$$\alpha\begin{bmatrix}1 & 3 & 4\\ 2 & 1 & -1\end{bmatrix}+\begin{bmatrix}4 & 3 & -6\\ 0 & 1 & 1\end{bmatrix}=\begin{bmatrix}7 & 12 & 6\\ 6 & 4 & -2\end{bmatrix}$$

Solution (Chris Black) The given equation

$$\begin{aligned}\begin{bmatrix}7 & 12 & 6\\ 6 & 4 & -2\end{bmatrix}&=\alpha\begin{bmatrix}1 & 3 & 4\\ 2 & 1 & -1\end{bmatrix}+\begin{bmatrix}4 & 3 & -6\\ 0 & 1 & 1\end{bmatrix}\\ &=\begin{bmatrix}\alpha & 3\alpha & 4\alpha\\ 2\alpha & \alpha & -\alpha\end{bmatrix}+\begin{bmatrix}4 & 3 & -6\\ 0 & 1 & 1\end{bmatrix}\\ &=\begin{bmatrix}4+\alpha & 3+3\alpha & -6+4\alpha\\ 2\alpha & 1+\alpha & 1-\alpha\end{bmatrix}\end{aligned}$$

leads to the 6 equations in α:

$$\begin{aligned}4+\alpha&=7\\ 3+3\alpha&=12\\ -6+4\alpha&=6\\ 2\alpha&=6\\ 1+\alpha&=4\\ 1-\alpha&=-2.\end{aligned}$$

The only value that solves all 6 equations is $\alpha=3$, which is the solution to the original matrix equation.

C13 (Chris Black) Solve the given matrix equation for α, or explain why no solution exists:

$$\alpha\begin{bmatrix}3 & 1\\ 2 & 0\\ 1 & 4\end{bmatrix}-\begin{bmatrix}4 & 1\\ 3 & 2\\ 0 & 1\end{bmatrix}=\begin{bmatrix}2 & 1\\ 1 & -2\\ 2 & 6\end{bmatrix}$$

Solution (Chris Black) The given equation

$$\begin{bmatrix}2 & 1\\ 1 & -2\\ 2 & 6\end{bmatrix}=\alpha\begin{bmatrix}3 & 1\\ 2 & 0\\ 1 & 4\end{bmatrix}+\begin{bmatrix}4 & 1\\ 3 & 2\\ 0 & 1\end{bmatrix}=\begin{bmatrix}3\alpha-4 & \alpha-1\\ 2\alpha-3 & -2\\ \alpha & 4\alpha-1\end{bmatrix}$$

gives a system of six equations in α:

$$\begin{aligned}3\alpha-4&=2\\ \alpha-1&=1\\ 2\alpha-3&=1\\ -2&=-2\\ \alpha&=2\\ 4\alpha-1&=6.\end{aligned}$$

Solving each of these equations, we see that the first three and the fifth all lead to the solution $\alpha=2$, the fourth equation is true no matter what the value of α, but the last equation is only solved by $\alpha=7/4$. Thus, the system has no solution, and the original matrix equation also has no solution.

C14 (Chris Black) Find α and β that solve the following equation:

$$\alpha\begin{bmatrix}1 & 2\\ 4 & 1\end{bmatrix}+\beta\begin{bmatrix}2 & 1\\ 3 & 1\end{bmatrix}=\begin{bmatrix}-1 & 4\\ 6 & 1\end{bmatrix}$$

Solution (Chris Black) The given equation

$$\begin{bmatrix}-1 & 4\\ 6 & 1\end{bmatrix}=\alpha\begin{bmatrix}1 & 2\\ 4 & 1\end{bmatrix}+\beta\begin{bmatrix}2 & 1\\ 3 & 1\end{bmatrix}=\begin{bmatrix}\alpha+2\beta & 2\alpha+\beta\\ 4\alpha+3\beta & \alpha+\beta\end{bmatrix}$$

gives a system of four equations in two variables

$$\begin{aligned}\alpha+2\beta&=-1\\ 2\alpha+\beta&=4\\ 4\alpha+3\beta&=6\end{aligned}$$

$$\alpha + \beta = 1.$$

Solving this linear system by row-reducing the augmnented matrix shows that $\alpha = 3$, $\beta = -2$ is the only solution.

In Chapter V we defined the operations of vector addition and vector scalar multiplication in Definition CVA and Definition CVSM. These two operations formed the underpinnings of the remainder of the chapter. We have now defined similar operations for matrices in Definition MA and Definition MSM. You will have noticed the resulting similarities between Theorem VSPCV and Theorem VSPM.
In Exercises M20–M25, you will be asked to extend these similarities to other fundamental definitions and concepts we first saw in Chapter V. This sequence of problems was suggested by Martin Jackson.

M20 (Robert Beezer) Suppose $S = \{B_1, B_2, B_3, \ldots, B_p\}$ is a set of matrices from M_{mn}. Formulate appropriate definitions for the following terms and give an example of the use of each.

1. A linear combination of elements of S.
2. A relation of linear dependence on S, both trivial and non-trivial.
3. S is a linearly independent set.
4. $\langle S \rangle$.

M21 (Robert Beezer) Show that the set S is linearly independent in $M_{2,2}$.

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Solution (Chris Black) Suppose there exist constants α, β, γ, and δ so that

$$\alpha \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \delta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then,

$$\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \delta \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so that $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. The only solution is then $\alpha = 0$, $\beta = 0$, $\gamma = 0$, and $\delta = 0$, so that the set S is a linearly independent set of matrices.

M22 (Robert Beezer) Determine if the set S below is linearly independent in $M_{2,3}$.

$$\left\{ \begin{bmatrix} -2 & 3 & 4 \\ -1 & 3 & -2 \end{bmatrix}, \begin{bmatrix} 4 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -2 & -2 \\ 2 & 2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 2 & -2 \\ 0 & -1 & -2 \end{bmatrix} \right\}$$

Solution (Chris Black) Suppose that there exist constants a_1, a_2, a_3, a_4, and a_5 so that

$$a_1 \begin{bmatrix} -2 & 3 & 4 \\ -1 & 3 & -2 \end{bmatrix} + a_2 \begin{bmatrix} 4 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix} + a_3 \begin{bmatrix} -1 & -2 & -2 \\ 2 & 2 & 2 \end{bmatrix} + a_4 \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} + a_5 \begin{bmatrix} -1 & 2 & -1 \\ 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, we have the matrix equality (Definition ME)

$$\begin{bmatrix} -2a_1 + 4a_2 - a_3 - a_4 - a_5 & 3a_1 - 2a_2 - 2a_3 + a_4 + 2a_5 & 4a_1 + 2a_2 - 2a_3 - 2a_5 \\ -a_1 + 2a_3 - a_4 & 3a_1 - a_2 + 2a_3 - a_5 & -2a_1 + a_2 + 2a_3 + 2a_4 - 2a_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which yields the linear system of equations

$$\begin{aligned} -2a_1 + 4a_2 - a_3 - a_4 - a_5 &= 0 \\ 3a_1 - 2a_2 - 2a_3 + a_4 + 2a_5 &= 0 \\ 4a_1 + 2a_2 - 2a_3 - 2a_5 &= 0 \\ -a_1 + 2a_3 - a_4 &= 0 \\ 3a_1 - a_2 + 2a_3 - a_5 &= 0 \\ -2a_1 + a_2 + 2a_3 + 2a_4 - 2a_5 &= 0. \end{aligned}$$

By row-reducing the associated 6×5 homogeneous system, we see that the only solution is $a_1 = a_2 = a_3 = a_4 = a_5 = 0$, so these matrices are a linearly independent subset of $M_{2,3}$.

M23 (Robert Beezer) Determine if the matrix A is in the span of S. In other words, is $A \in \langle S \rangle$? If so write A as a linear combination of the elements of S.

$$A = \begin{bmatrix} -13 & 24 & 2 \\ -8 & -2 & -20 \end{bmatrix}$$
$$S = \left\{ \begin{bmatrix} -2 & 3 & 4 \\ -1 & 3 & -2 \end{bmatrix}, \begin{bmatrix} 4 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -2 & -2 \\ 2 & 2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 2 & -2 \\ 0 & -1 & -2 \end{bmatrix} \right\}$$

Solution (Chris Black) The matrix A is in the span of S, since

$$\begin{bmatrix} -13 & 24 & 2 \\ -8 & -2 & -20 \end{bmatrix} = 2\begin{bmatrix} -2 & 3 & 4 \\ -1 & 3 & -2 \end{bmatrix} - 2\begin{bmatrix} 4 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix} - 3\begin{bmatrix} -1 & -2 & -2 \\ 2 & 2 & 2 \end{bmatrix} + 4\begin{bmatrix} -1 & 2 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$

Note that if we were to write a complete linear combination of *all* of the matrices in S, then the fourth matrix would have a zero coefficient.

M24 (Robert Beezer) Suppose Y is the set of all 3×3 symmetric matrices (Definition SYM). Find a set T so that T is linearly independent and $\langle T \rangle = Y$.

Solution (Chris Black) Since any symmetric matrix is of the form

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ c & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & e \\ 0 & e & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f \end{bmatrix},$$

Any symmetric matrix is a linear combination of the linearly independent vectors in set T below, so that $\langle T \rangle = Y$:

$$T = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

(Something to think about: How do we know that these matrices are linearly independent?)

M25 (Robert Beezer) Define a subset of $M_{3,3}$ by

$$U_{33} = \left\{ A \in M_{3,3} \mid [A]_{ij} = 0 \text{ whenever } i > j \right\}$$

Find a set R so that R is linearly independent and $\langle R \rangle = U_{33}$.

T13 (Robert Beezer) Prove Property CM of Theorem VSPM. Write your proof in the style of the proof of Property DSAM given in this section.

Solution (Robert Beezer) For all A, $B \in M_{mn}$ and for all $1 \leq i \leq m$, $1 \leq i \leq n$,

$$\begin{aligned} [A + B]_{ij} &= [A]_{ij} + [B]_{ij} && \text{Definition MA} \\ &= [B]_{ij} + [A]_{ij} && \text{Commutativity in } \mathbb{C} \\ &= [B + A]_{ij} && \text{Definition MA} \end{aligned}$$

With equality of each entry of the matrices $A + B$ and $B + A$ being equal Definition ME tells us the two matrices are equal.

T14 (Robert Beezer) Prove Property AAM of Theorem VSPM. Write your proof in the style of the proof of Property DSAM given in this section.

T17 (Robert Beezer) Prove Property SMAM of Theorem VSPM. Write your proof in the style of the proof of Property DSAM given in this section.

T18 (Robert Beezer) Prove Property DMAM of Theorem VSPM. Write your proof in the style of the proof of Property DSAM given in this section.

A matrix A is **skew-symmetric** if $A^t = -A$ Exercises T30–T37 employ this definition.

T30 (Robert Beezer) Prove that a skew-symmetric matrix is square. (Hint: study the proof of Theorem SMS.)

T31 (Manley Perkel) Prove that a skew-symmetric matrix must have zeros for its diagonal elements. In other words, if A is skew-symmetric of size n, then $[A]_{ii} = 0$ for $1 \leq i \leq n$. (Hint: carefully construct an example of a 3×3 skew-symmetric matrix before attempting a proof.)

T32 (Manley Perkel) Prove that a matrix A is both skew-symmetric and symmetric if and only if A is the zero matrix. (Hint: one half of this proof is very easy, the other half takes a little more work.)

T33 (Manley Perkel) Suppose A and B are both skew-symmetric matrices of the same size and $\alpha,\ \beta \in \mathbb{C}$. Prove that $\alpha A + \beta B$ is a skew-symmetric matrix.

T34 (Manley Perkel) Suppose A is a square matrix. Prove that $A + A^t$ is a symmetric matrix.

T35 (Manley Perkel) Suppose A is a square matrix. Prove that $A - A^t$ is a skew-symmetric matrix.

T36 (Manley Perkel) Suppose A is a square matrix. Prove that there is a symmetric matrix B and a skew-symmetric matrix C such that $A = B + C$. In other words, any square matrix can be decomposed into a symmetric matrix and a skew-symmetric matrix (Proof Technique DC). (Hint: consider building a proof on Exercise MO.T34 and Exercise MO.T35.)

T37 (Manley Perkel) Prove that the decomposition in Exercise MO.T36 is unique (see Proof Technique U). (Hint: a proof can turn on Exercise MO.T31.)

Section MM
Matrix Multiplication

C20 (Robert Beezer) Compute the product of the two matrices below, AB. Do this using the definitions of the matrix-vector product (Definition MVP) and the definition of matrix multiplication (Definition MM).

$$A = \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 5 & -3 & 4 \\ 2 & 0 & 2 & -3 \end{bmatrix}$$

Solution (Robert Beezer) By Definition MM,

$$AB = \left[\begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \middle| \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix} \middle| \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix} \middle| \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix} \right]$$

Repeated applications of Definition MVP give

$$= \left[1\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} \middle| 5\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} \middle| -3\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} \middle| 4\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + (-3)\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 12 & 10 & 4 & -7 \\ 5 & -5 & 9 & -13 \\ -2 & 10 & -10 & 14 \end{bmatrix}$$

C21 (Chris Black) Compute the product AB of the two matrices below using both the definition of the matrix-vector product (Definition MVP) and the definition of matrix multiplication (Definition MM).

$$A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 0 & 1 \\ 3 & 1 & 5 \end{bmatrix}$$

Solution (Chris Black) $AB = \begin{bmatrix} 13 & 3 & 15 \\ 1 & 0 & 5 \\ 1 & 0 & 1 \end{bmatrix}$.

C22 (Chris Black) Compute the product AB of the two matrices below using both the definition of the matrix-vector product (Definition MVP) and the definition of matrix multiplication (Definition MM).

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$$

Solution (Chris Black) $AB = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$.

C23 (Chris Black) Compute the product AB of the two matrices below using both the definition of the matrix-vector product (Definition MVP) and the definition of matrix multiplication (Definition MM).

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ 6 & 5 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$$

Solution (Chris Black) $AB = \begin{bmatrix} -5 & 5 \\ 10 & 10 \\ 2 & 16 \\ 5 & 5 \end{bmatrix}$.

C24 (Chris Black) Compute the product AB of the two matrices below.

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & 1 & -2 & -1 \\ 1 & 1 & 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 2 \end{bmatrix}$$

Solution (Chris Black) $AB = \begin{bmatrix} 7 \\ 2 \\ 9 \end{bmatrix}$.

C25 (Chris Black) Compute the product AB of the two matrices below.

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & 1 & -2 & -1 \\ 1 & 1 & 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -7 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

Solution (Chris Black) $AB = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

C26 (Chris Black) Compute the product AB of the two matrices below using both the definition of the matrix-vector product (Definition MVP) and the definition of matrix multiplication (Definition MM).

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -5 & -1 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

Solution (Chris Black) $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

C30 (Chris Black) For the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, find A^2, A^3, A^4. Find a general formula for A^n for any positive integer n.

Solution (Chris Black) $A^2 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$, $A^4 = \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}$. From this pattern, we see that $A^n = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}$.

C31 (Chris Black) For the matrix $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, find A^2, A^3, A^4. Find a general formula for A^n for any positive integer n.

Solution (Chris Black) $A^2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$, $A^4 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$. From this pattern, we see that $A^n = \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix}$.

C32 (Chris Black) For the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, find A^2, A^3, A^4. Find a general formula for A^n for any positive integer n.

Solution (Chris Black) $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 27 \end{bmatrix}$, and $A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}$. The pattern emerges, and we see that $A^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix}$.

C33 (Chris Black) For the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, find A^2, A^3, A^4. Find a general formula for A^n for any positive integer n.

Solution (Chris Black) We quickly compute $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and we then see that A^3 and all subsequent powers of A are the 3×3 zero matrix; that is, $A^n = \mathcal{O}_{3,3}$ for $n \geq 3$.

T10 (Robert Beezer) Suppose that A is a square matrix and there is a vector, $\mathbf{b}$, such that $\mathcal{LS}(A, \mathbf{b})$ has a unique solution. Prove that A is nonsingular. Give a direct proof (perhaps appealing to Theorem PSPHS) rather than just negating a sentence from the text discussing a similar situation.

Solution (Robert Beezer) Since $\mathcal{LS}(A, \mathbf{b})$ has at least one solution, we can apply Theorem PSPHS. Because the solution is assumed to be unique, the null space of A must be trivial. Then Theorem NMTNS implies that A is nonsingular.

The converse of this statement is a trivial application of Theorem NMUS. That said, we could extend our NSMxx series of theorems with an added equivalence for nonsingularity, "Given a single vector of constants, $\mathbf{b}$, the system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution."

T12 (Robert Beezer) The conclusion of Theorem HMIP is $\langle A\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, A^*\mathbf{y}\rangle$. Use the same hypotheses, and prove the similar conclusion: $\langle \mathbf{x}, A\mathbf{y}\rangle = \langle A^*\mathbf{x}, \mathbf{y}\rangle$. Two different approaches can be based on an application of Theorem HMIP. The first uses Theorem AA, while a second uses Theorem IPAC. Can you provide two proofs?

T20 (Robert Beezer) Prove the second part of Theorem MMZM.

T21 (Robert Beezer) Prove the second part of Theorem MMIM.

T22 (Robert Beezer) Prove the second part of Theorem MMDAA.

T23 (Robert Beezer) Prove the second part of Theorem MMSMM.

Solution (Robert Beezer) We'll run the proof entry-by-entry.

$$\begin{aligned} [\alpha(AB)]_{ij} &= \alpha\,[AB]_{ij} && \text{Definition MSM} \\ &= \alpha \sum_{k=1}^{n} [A]_{ik}\,[B]_{kj} && \text{Theorem EMP} \end{aligned}$$

$$\begin{aligned} &= \sum_{k=1}^{n} \alpha\, [A]_{ik}\, [B]_{kj} && \text{Distributivity in } \mathbb{C} \\ &= \sum_{k=1}^{n} [A]_{ik}\, \alpha\, [B]_{kj} && \text{Commutativity in } \mathbb{C} \\ &= \sum_{k=1}^{n} [A]_{ik}\, [\alpha B]_{kj} && \text{Definition MSM} \\ &= [A(\alpha B)]_{ij} && \text{Theorem EMP} \end{aligned}$$

So the matrices $\alpha(AB)$ and $A(\alpha B)$ are equal, entry-by-entry, and by the definition of matrix equality (Definition ME) we can say they are equal matrices.

T31 (Robert Beezer) Suppose that A is an $m \times n$ matrix and $\mathbf{x},\, \mathbf{y} \in \mathcal{N}(A)$. Prove that $\mathbf{x} + \mathbf{y} \in \mathcal{N}(A)$.

T32 (Robert Beezer) Suppose that A is an $m \times n$ matrix, $\alpha \in \mathbb{C}$, and $\mathbf{x} \in \mathcal{N}(A)$. Prove that $\alpha\mathbf{x} \in \mathcal{N}(A)$.

T35 (Robert Beezer) Suppose that A is an $n \times n$ matrix. Prove that A^*A and AA^* are Hermitian matrices.

T40 (Robert Beezer) Suppose that A is an $m \times n$ matrix and B is an $n \times p$ matrix. Prove that the null space of B is a subset of the null space of AB, that is $\mathcal{N}(B) \subseteq \mathcal{N}(AB)$. Provide an example where the opposite is false, in other words give an example where $\mathcal{N}(AB) \not\subseteq \mathcal{N}(B)$.

Solution (Robert Beezer) To prove that one set is a subset of another, we start with an element of the smaller set and see if we can determine that it is a member of the larger set (Definition SSET). Suppose $\mathbf{x} \in \mathcal{N}(B)$. Then we know that $B\mathbf{x} = \mathbf{0}$ by Definition NSM. Consider

$$\begin{aligned} (AB)\mathbf{x} &= A(B\mathbf{x}) && \text{Theorem MMA} \\ &= A\mathbf{0} && \text{Hypothesis} \\ &= \mathbf{0} && \text{Theorem MMZM} \end{aligned}$$

This establishes that $\mathbf{x} \in \mathcal{N}(AB)$, so $\mathcal{N}(B) \subseteq \mathcal{N}(AB)$.

To show that the inclusion does not hold in the opposite direction, choose B to be any nonsingular matrix of size n. Then $\mathcal{N}(B) = \{\mathbf{0}\}$ by Theorem NMTNS. Let A be the square zero matrix, $\mathcal{O}$, of the same size. Then $AB = \mathcal{O}B = \mathcal{O}$ by Theorem MMZM and therefore $\mathcal{N}(AB) = \mathbb{C}^n$, and is *not* a subset of $\mathcal{N}(B) = \{\mathbf{0}\}$.

T41 (Robert Beezer) Suppose that A is an $n \times n$ nonsingular matrix and B is an $n \times p$ matrix. Prove that the null space of B is equal to the null space of AB, that is $\mathcal{N}(B) = \mathcal{N}(AB)$. (Compare with Exercise MM.T40.)

Solution (David Braithwaite) From the solution to Exercise MM.T40 we know that $\mathcal{N}(B) \subseteq \mathcal{N}(AB)$. So to establish the set equality (Definition SE) we need to show that $\mathcal{N}(AB) \subseteq \mathcal{N}(B)$.

Suppose $\mathbf{x} \in \mathcal{N}(AB)$. Then we know that $AB\mathbf{x} = \mathbf{0}$ by Definition NSM. Consider

$$\begin{aligned} \mathbf{0} &= (AB)\,\mathbf{x} && \text{Definition NSM} \\ &= A\,(B\mathbf{x}) && \text{Theorem MMA} \end{aligned}$$

So, $B\mathbf{x} \in \mathcal{N}(A)$. Because A is nonsingular, it has a trivial null space (Theorem NMTNS) and we conclude that $B\mathbf{x} = \mathbf{0}$. This establishes that $\mathbf{x} \in \mathcal{N}(B)$, so $\mathcal{N}(AB) \subseteq \mathcal{N}(B)$ and combined with the solution to Exercise MM.T40 we have $\mathcal{N}(B) = \mathcal{N}(AB)$ when A is nonsingular.

T50 (Robert Beezer) Suppose $\mathbf{u}$ and $\mathbf{v}$ are any two solutions of the linear system $\mathcal{LS}(A,\, \mathbf{b})$. Prove that $\mathbf{u} - \mathbf{v}$ is an element of the null space of A, that is, $\mathbf{u} - \mathbf{v} \in \mathcal{N}(A)$.

T51 (Robert Beezer) Give a new proof of Theorem PSPHS replacing applications of Theorem SLSLC with matrix-vector products (Theorem SLEMM).

Solution (Robert Beezer) We will work with the vector equality representations of the relevant systems of equations, as described by Theorem SLEMM.

($\Leftarrow$) Suppose $\mathbf{y} = \mathbf{w} + \mathbf{z}$ and $\mathbf{z} \in \mathcal{N}(A)$. Then

$$\begin{aligned} A\mathbf{y} &= A(\mathbf{w} + \mathbf{z}) && \text{Substitution} \\ &= A\mathbf{w} + A\mathbf{z} && \text{Theorem MMDAA} \\ &= \mathbf{b} + \mathbf{0} && \mathbf{z} \in \mathcal{N}(A) \\ &= \mathbf{b} && \text{Property ZC} \end{aligned}$$

demonstrating that $\mathbf{y}$ is a solution.

($\Rightarrow$) Suppose $\mathbf{y}$ is a solution to $\mathcal{LS}(A,\,\mathbf{b})$. Then

$$\begin{aligned} A(\mathbf{y} - \mathbf{w}) &= A\mathbf{y} - A\mathbf{w} && \text{Theorem MMDAA} \\ &= \mathbf{b} - \mathbf{b} && \mathbf{y},\,\mathbf{w} \text{ solutions to } A\mathbf{x} = \mathbf{b} \\ &= \mathbf{0} && \text{Property AIC} \end{aligned}$$

which says that $\mathbf{y} - \mathbf{w} \in \mathcal{N}(A)$. In other words, $\mathbf{y} - \mathbf{w} = \mathbf{z}$ for some vector $\mathbf{z} \in \mathcal{N}(A)$. Rewritten, this is $\mathbf{y} = \mathbf{w} + \mathbf{z}$, as desired.

T52 (Robert Beezer) Suppose that $\mathbf{x},\,\mathbf{y} \in \mathbb{C}^n$, $\mathbf{b} \in \mathbb{C}^m$ and A is an $m \times n$ matrix. If $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{x} + \mathbf{y}$ are each a solution to the linear system $\mathcal{LS}(A,\,\mathbf{b})$, what interesting can you say about $\mathbf{b}$? Form an implication with the existence of the three solutions as the hypothesis and an interesting statement about $\mathcal{LS}(A,\,\mathbf{b})$ as the conclusion, and then give a proof.

Solution (Robert Beezer) $\mathcal{LS}(A,\,\mathbf{b})$ must be homogeneous. To see this consider that

$$\begin{aligned} \mathbf{b} &= A\mathbf{x} && \text{Theorem SLEMM} \\ &= A\mathbf{x} + \mathbf{0} && \text{Property ZC} \\ &= A\mathbf{x} + A\mathbf{y} - A\mathbf{y} && \text{Property AIC} \\ &= A\left(\mathbf{x} + \mathbf{y}\right) - A\mathbf{y} && \text{Theorem MMDAA} \\ &= \mathbf{b} - \mathbf{b} && \text{Theorem SLEMM} \\ &= \mathbf{0} && \text{Property AIC} \end{aligned}$$

By Definition HS we see that $\mathcal{LS}(A,\,\mathbf{b})$ is homogeneous.

Section MISLE
Matrix Inverses and Systems of Linear Equations

C16 (Chris Black) If it exists, find the inverse of $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}$, and check your answer.

Solution (Chris Black) Answer: $A^{-1} = \begin{bmatrix} -2 & 1 & 1 \\ -1 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix}$.

C17 (Chris Black) If it exists, find the inverse of $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, and check your answer.

Solution (Chris Black) The procedure we have for finding a matrix inverse fails for this matrix A since A does not row-reduce to I_3. We suspect in this case that A is not invertible, although we do not yet know that concretely. (Stay tuned for upcoming revelations in Section MINM!)

C18 (Chris Black) If it exists, find the inverse of $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & 2 & 1 \end{bmatrix}$, and check your answer.

Solution (Chris Black) Answer: $A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \\ -2 & 4 & -1 \end{bmatrix}$

C19 (Chris Black) If it exists, find the inverse of $A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 1 \\ 2 & 2 & 1 \end{bmatrix}$, and check your answer.

Solution (Chris Black) Answer: $A^{-1} = \begin{bmatrix} 0 & -1/2 & 1/2 \\ 1 & -1/2 & -1/2 \\ -2 & 2 & 1 \end{bmatrix}$

C21 (Robert Beezer) Verify that B is the inverse of A.

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ -2 & -1 & 2 & -3 \\ 1 & 1 & 0 & 2 \\ -1 & 2 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 2 & 0 & -1 \\ 8 & 4 & -1 & -1 \\ -1 & 0 & 1 & 0 \\ -6 & -3 & 1 & 1 \end{bmatrix}$$

Solution (Robert Beezer) Check that *both* matrix products (Definition MM) AB and BA equal the 4×4 identity matrix I_4 (Definition IM).

C22 (Robert Beezer) Recycle the matrices A and B from Exercise MISLE.C21 and set

$$\mathbf{c} = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 2 \end{bmatrix} \qquad \mathbf{d} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Employ the matrix B to solve the two linear systems $\mathcal{LS}(A,\,\mathbf{c})$ and $\mathcal{LS}(A,\,\mathbf{d})$.

Solution (Robert Beezer) Represent each of the two systems by a vector equality, $A\mathbf{x} = \mathbf{c}$ and $A\mathbf{y} = \mathbf{d}$. Then in the spirit of Example SABMI, solutions are given by

$$\mathbf{x} = B\mathbf{c} = \begin{bmatrix} 8 \\ 21 \\ -5 \\ -16 \end{bmatrix} \qquad \mathbf{y} = B\mathbf{d} = \begin{bmatrix} 5 \\ 10 \\ 0 \\ -7 \end{bmatrix}$$

Notice how we could solve many more systems having A as the coefficient matrix, and how each such system has a unique solution. You might check your work by substituting the solutions back into the systems of equations, or forming the linear combinations of the columns of A suggested by Theorem SLSLC.

C23 (Robert Beezer) If it exists, find the inverse of the 2×2 matrix

$$A = \begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix}$$

and check your answer. (See Theorem TTMI.)

C24 (Robert Beezer) If it exists, find the inverse of the 2×2 matrix

$$A = \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}$$

and check your answer. (See Theorem TTMI.)

C25 (Robert Beezer) At the conclusion of Example CMI, verify that $BA = I_5$ by computing the matrix product.

C26 (Robert Beezer) Let

$$D = \begin{bmatrix} 1 & -1 & 3 & -2 & 1 \\ -2 & 3 & -5 & 3 & 0 \\ 1 & -1 & 4 & -2 & 2 \\ -1 & 4 & -1 & 0 & 4 \\ 1 & 0 & 5 & -2 & 5 \end{bmatrix}$$

Compute the inverse of D, D^{-1}, by forming the 5×10 matrix $[D \mid I_5]$ and row-reducing (Theorem CINM). Then use a calculator to compute D^{-1} directly.

Solution (Robert Beezer) The inverse of D is

$$D^{-1} = \begin{bmatrix} -7 & -6 & -3 & 2 & 1 \\ -7 & -4 & 2 & 2 & -1 \\ -5 & -2 & 3 & 1 & -1 \\ -6 & -3 & 1 & 1 & 0 \\ 4 & 2 & -2 & -1 & 1 \end{bmatrix}$$

C27 (Robert Beezer) Let

$$E = \begin{bmatrix} 1 & -1 & 3 & -2 & 1 \\ -2 & 3 & -5 & 3 & -1 \\ 1 & -1 & 4 & -2 & 2 \\ -1 & 4 & -1 & 0 & 2 \\ 1 & 0 & 5 & -2 & 4 \end{bmatrix}$$

Compute the inverse of E, E^{-1}, by forming the 5×10 matrix $[E \mid I_5]$ and row-reducing (Theorem CINM). Then use a calculator to compute E^{-1} directly.

Solution (Robert Beezer) The matrix E has no inverse, though we do not yet have a theorem that allows us to reach this conclusion. However, when row-reducing the matrix $[E \mid I_5]$, the first 5 columns will not row-reduce to the 5×5 identity matrix, so we are a t a loss on how we might compute the inverse. When requesting that your calculator compute E^{-1}, it should give some indication that E does not have an inverse.

C28 (Robert Beezer) Let

$$C = \begin{bmatrix} 1 & 1 & 3 & 1 \\ -2 & -1 & -4 & -1 \\ 1 & 4 & 10 & 2 \\ -2 & 0 & -4 & 5 \end{bmatrix}$$

Compute the inverse of C, C^{-1}, by forming the 4×8 matrix $[C \mid I_4]$ and row-reducing (Theorem CINM). Then use a calculator to compute C^{-1} directly.

Solution (Robert Beezer) Employ Theorem CINM,

$$\begin{bmatrix} 1 & 1 & 3 & 1 & 1 & 0 & 0 & 0 \\ -2 & -1 & -4 & -1 & 0 & 1 & 0 & 0 \\ 1 & 4 & 10 & 2 & 0 & 0 & 1 & 0 \\ -2 & 0 & -4 & 5 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 38 & 18 & -5 & -2 \\ 0 & \boxed{1} & 0 & 0 & 96 & 47 & -12 & -5 \\ 0 & 0 & \boxed{1} & 0 & -39 & -19 & 5 & 2 \\ 0 & 0 & 0 & \boxed{1} & -16 & -8 & 2 & 1 \end{bmatrix}$$

And therefore we see that C is nonsingular (C row-reduces to the identity matrix, Theorem NMRRI) and by Theorem CINM,

$$C^{-1} = \begin{bmatrix} 38 & 18 & -5 & -2 \\ 96 & 47 & -12 & -5 \\ -39 & -19 & 5 & 2 \\ -16 & -8 & 2 & 1 \end{bmatrix}$$

C40 (Robert Beezer) Find all solutions to the system of equations below, making use of the matrix inverse found in Exercise MISLE.C28.

$$\begin{aligned} x_1 + x_2 + 3x_3 + x_4 &= -4 \\ -2x_1 - x_2 - 4x_3 - x_4 &= 4 \\ x_1 + 4x_2 + 10x_3 + 2x_4 &= -20 \\ -2x_1 - 4x_3 + 5x_4 &= 9 \end{aligned}$$

Solution (Robert Beezer) View this system as $\mathcal{LS}(C, \mathbf{b})$, where C is the 4×4 matrix from Exercise MISLE.C28 and $\mathbf{b} = \begin{bmatrix} -4 \\ 4 \\ -20 \\ 9 \end{bmatrix}$. Since C was seen to be nonsingular in Exercise MISLE.C28 Theorem SNCM says the solution, which is unique by Theorem NMUS, is given by

$$C^{-1}\mathbf{b} = \begin{bmatrix} 38 & 18 & -5 & -2 \\ 96 & 47 & -12 & -5 \\ -39 & -19 & 5 & 2 \\ -16 & -8 & 2 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 4 \\ -20 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \\ 1 \end{bmatrix}$$

Notice that this solution can be easily checked in the original system of equations.

C41 (Robert Beezer) Use the inverse of a matrix to find all the solutions to the following system of equations.

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -3 \\ 2x_1 + 5x_2 - x_3 &= -4 \\ -x_1 - 4x_2 &= 2 \end{aligned}$$

Solution (Robert Beezer) The coefficient matrix of this system of equations is

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ -1 & -4 & 0 \end{bmatrix}$$

and the vector of constants is $\mathbf{b} = \begin{bmatrix} -3 \\ -4 \\ 2 \end{bmatrix}$. So by Theorem SLEMM we can convert the system to the form $A\mathbf{x} = \mathbf{b}$. Row-reducing this matrix yields the identity matrix so by Theorem NMRRI we know A is nonsingular. This allows us to apply Theorem SNCM to find the unique solution as

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -4 & 4 & 3 \\ 1 & -1 & -1 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Remember, you can check this solution easily by evaluating the matrix-vector product $A\mathbf{x}$ (Definition MVP).

C42 (Robert Beezer) Use a matrix inverse to solve the linear system of equations.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 5 \\ x_1 - 2x_3 &= -8 \\ 2x_1 - x_2 - x_3 &= -6 \end{aligned}$$

Solution (Robert Beezer) We can reformulate the linear system as a vector equality with a matrix-vector product via Theorem SLEMM. The system is then represented by $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 5 \\ -8 \\ -6 \end{bmatrix}$$

According to Theorem SNCM, if A is nonsingular then the (unique) solution will be given by $A^{-1}\mathbf{b}$. We attempt the computation of A^{-1} through Theorem CINM, or with our favorite computational device and obtain,

$$A^{-1} = \begin{bmatrix} 2 & 3 & -2 \\ 3 & 5 & -4 \\ 1 & 1 & -1 \end{bmatrix}$$

So by Theorem NI, we know A is nonsingular, and so the unique solution is

$$A^{-1}\mathbf{b} = \begin{bmatrix} 2 & 3 & -2 \\ 3 & 5 & -4 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ -8 \\ -6 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$$

T10 (Robert Beezer) Construct an example to demonstrate that $(A + B)^{-1} = A^{-1} + B^{-1}$ is not true for all square matrices A and B of the same size.

Solution (Robert Beezer) For a large collection of small examples, let D be any 2×2 matrix that has an inverse (Theorem TTMI can help you construct such a matrix, I_2 is a simple choice). Set $A = D$ and $B = (-1)D$. While A^{-1} and B^{-1} both exist, what is $(A + B)^{-1}$?

For a large collection of examples of any size, consider $A = B = I_n$. Can the proposed statement be salvaged to become a theorem?

Section MINM
Matrix Inverses and Nonsingular Matrices

C20 (Chris Black) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Verify that AB is nonsingular.

C40 (Robert Beezer) Solve the system of equations below using the inverse of a matrix.

$$\begin{aligned} x_1 + x_2 + 3x_3 + x_4 &= 5 \\ -2x_1 - x_2 - 4x_3 - x_4 &= -7 \\ x_1 + 4x_2 + 10x_3 + 2x_4 &= 9 \\ -2x_1 - 4x_3 + 5x_4 &= 9 \end{aligned}$$

Solution (Robert Beezer) The coefficient matrix and vector of constants for the system are

$$\begin{bmatrix} 1 & 1 & 3 & 1 \\ -2 & -1 & -4 & -1 \\ 1 & 4 & 10 & 2 \\ -2 & 0 & -4 & 5 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 5 \\ -7 \\ 9 \\ 9 \end{bmatrix}$$

A^{-1} can be computed by using a calculator, or by the method of Theorem CINM. Then Theorem SNCM says the unique solution is

$$A^{-1}\mathbf{b} = \begin{bmatrix} 38 & 18 & -5 & -2 \\ 96 & 47 & -12 & -5 \\ -39 & -19 & 5 & 2 \\ -16 & -8 & 2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -7 \\ 9 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 3 \end{bmatrix}$$

M10 (Chris Black) Find values of x, y, z so that matrix $A = \begin{bmatrix} 1 & 2 & x \\ 3 & 0 & y \\ 1 & 1 & z \end{bmatrix}$ is invertible.

Solution (Chris Black) There are an infinite number of possible answers. We want to find a vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ so that the set

$$S = \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\}$$

is a linearly independent set. We need a vector not in the span of the first two columns, which geometrically means that we need it to not be in the same plane as the first two columns of A. We can choose any values we want for x and y, and then choose a value of z that makes the three vectors independent.

I will (arbitrarily) choose $x = 1$, $y = 1$. Then, we have

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 1 & 1 & z \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2z-1 \\ 0 & \boxed{1} & 1-z \\ 0 & 0 & 4-6z \end{bmatrix}$$

which is invertible if and only if $4 - 6z \neq 0$. Thus, we can choose any value as long as $z \neq \frac{2}{3}$, so we choose $z = 0$, and we have found a matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ that is invertible.

M11 (Chris Black) Find values of x, y z so that matrix $A = \begin{bmatrix} 1 & x & 1 \\ 1 & y & 4 \\ 0 & z & 5 \end{bmatrix}$ is singular.

Solution (Chris Black) There are an infinite number of possible answers. We need the set of vectors

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \right\}$$

to be linearly dependent. One way to do this by inspection is to have $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$. Thus, if we let $x = 1$, $y = 4$, $z = 5$, then the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 0 & 5 & 5 \end{bmatrix}$ is singular.

M15 (Chris Black) If A and B are $n \times n$ matrices, A is nonsingular, and B is singular, show directly that AB is singular, without using Theorem NPNT.

Solution (Chris Black) If B is singular, then there exists a vector $\mathbf{x} \neq \mathbf{0}$ so that $\mathbf{x} \in \mathcal{N}(B)$. Thus, $B\mathbf{x} = \mathbf{0}$, so $A(B\mathbf{x}) = (AB)\mathbf{x} = \mathbf{0}$, so $\mathbf{x} \in \mathcal{N}(AB)$. Since the null space of AB is not trivial, AB is a singular matrix.

M20 (Robert Beezer) Construct an example of a 4×4 unitary matrix.

Solution (Robert Beezer) The 4×4 identity matrix, I_4, would be one example (Definition IM). Any of the 23 other rearrangements of the columns of I_4 would be a simple, but less trivial, example. See Example UPM.

M80 (Mark Hamrick) Matrix multiplication interacts nicely with many operations. But not always with transforming a matrix to reduced row-echelon form. Suppose that A is an $m \times n$ matrix and B is an $n \times p$ matrix. Let P be a matrix that is row-equivalent to A and in reduced row-echelon form, Q be a matrix that is row-equivalent to B and in reduced row-echelon form, and let R be a matrix that is row-equivalent to AB and in reduced row-echelon form. Is $PQ = R$? (In other words, with nonstandard notation, is $\operatorname{rref}(A)\operatorname{rref}(B) = \operatorname{rref}(AB)$?)

Construct a counterexample to show that, in general, this statement is false. Then find a large class of matrices where if A and B are in the class, then the statement is true.

Solution (Robert Beezer) Take

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Then A is already in reduced row-echelon form, and by swapping rows, B row-reduces to A. So the product of the row-echelon forms of A is $AA = A \neq \mathcal{O}$. However, the product AB is the 2×2 zero matrix, which is in reduced-echelon form, and not equal to AA. When you get there, Theorem PEEF or Theorem EMDRO might shed some light on why we would not expect this statement to be true in general.

If A and B are nonsingular, then AB is nonsingular (Theorem NPNT), and all three matrices A, B and AB row-reduce to the identity matrix (Theorem NMRRI). By Theorem MMIM, the desired relationship is true.

T10 (Robert Beezer) Suppose that Q and P are unitary matrices of size n. Prove that QP is a unitary matrix.

T11 (Robert Beezer) Prove that Hermitian matrices (Definition HM) have real entries on the diagonal. More precisely, suppose that A is a Hermitian matrix of size n. Then $[A]_{ii} \in \mathbb{R}$, $1 \leq i \leq n$.

T12 (Robert Beezer) Suppose that we are checking if a square matrix of size n is unitary. Show that a straightforward application of Theorem CUMOS requires the computation of n^2 inner products when the matrix is unitary, and fewer when the matrix is not orthogonal. Then show that this maximum number of inner products can be reduced to $\frac{1}{2}n(n+1)$ in light of Theorem IPAC.

T25 (Manley Perkel) The notation A^k means a repeated matrix product between k copies of the square matrix A.

1. Assume A is an $n \times n$ matrix where $A^2 = \mathcal{O}$ (which does not imply that $A = \mathcal{O}$.) Prove that $I_n - A$ is invertible by showing that $I_n + A$ is an inverse of $I_n - A$.
2. Assume that A is an $n \times n$ matrix where $A^3 = \mathcal{O}$. Prove that $I_n - A$ is invertible.
3. Form a general theorem based on your observations from parts (1) and (2) and provide a proof.

Section CRS
Column and Row Spaces

C20 (Chris Black) For parts (1), (2) and (3), find a set of linearly independent vectors X so that $\mathcal{C}(A) = \langle X \rangle$, and a set of linearly independent vectors Y so that $\mathcal{R}(A) = \langle Y \rangle$.

1. $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & -1 & 2 & 3 \\ 1 & 1 & 2 & -1 \end{bmatrix}$
2. $A = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 3 & 2 & -1 & 4 & 5 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix}$
3. $A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 0 & 3 \\ 1 & 2 & -3 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$
4. From your results in parts (1) - (3), can you formulate a conjecture about the sets X and Y?

C30 (Robert Beezer) Example CSOCD expresses the column space of the coefficient matrix from Archetype D (call the matrix A here) as the span of the first two columns of A. In Example CSMCS we determined that the vector

$$\mathbf{c} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

was not in the column space of A and that the vector

$$\mathbf{b} = \begin{bmatrix} 8 \\ -12 \\ 4 \end{bmatrix}$$

was in the column space of A. Attempt to write $\mathbf{c}$ and $\mathbf{b}$ as linear combinations of the two vectors in the span construction for the column space in Example CSOCD and record your observations.

Solution (Robert Beezer) In each case, begin with a vector equation where one side contains a linear combination of the two vectors from the span construction that gives the column space of A with unknowns for scalars, and then use Theorem SLSLC to set up a system of equations. For $\mathbf{c}$, the corresponding system has no solution, as we would expect.

For $\mathbf{b}$ there is a solution, as we would expect. What is interesting is that the solution is unique. This is a consequence of the linear independence of the set of two vectors in the span construction. If we wrote $\mathbf{b}$ as a linear combination of all four columns of A, then there would be infinitely many ways to do this.

C31 (Robert Beezer) For the matrix A below find a set of vectors T meeting the following requirements: (1) the span of T is the column space of A, that is, $\langle T \rangle = \mathcal{C}(A)$, (2) T is linearly independent, and (3) the elements of T are columns of A.

$$A = \begin{bmatrix} 2 & 1 & 4 & -1 & 2 \\ 1 & -1 & 5 & 1 & 1 \\ -1 & 2 & -7 & 0 & 1 \\ 2 & -1 & 8 & -1 & 2 \end{bmatrix}$$

Solution (Robert Beezer) Theorem BCS is the right tool for this problem. Row-reduce this matrix, identify the pivot columns and then grab the corresponding columns of A for the set T. The matrix A row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 3 & 0 & 0 \\ 0 & \boxed{1} & -2 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

So $D = \{1,\, 2,\, 4,\, 5\}$ and then

$$T = \{\mathbf{A}_1,\, \mathbf{A}_2,\, \mathbf{A}_4,\, \mathbf{A}_5\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix},\, \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix},\, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix},\, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

has the requested properties.

C32 (Robert Beezer) In Example CSAA, verify that the vector $\mathbf{b}$ is not in the column space of the coefficient matrix.

C33 (Robert Beezer) Find a linearly independent set S so that the span of S, $\langle S\rangle$, is row space of the matrix B, and S is linearly independent.

$$B = \begin{bmatrix} 2 & 3 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & 2 & 3 & -4 \end{bmatrix}$$

Solution (Robert Beezer) Theorem BRS is the most direct route to a set with these properties. Row-reduce, toss zero rows, keep the others. You could also transpose the matrix, then look for the column space by row-reducing the transpose and applying Theorem BCS. We'll do the former,

$$B \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 & 2 \\ 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the set S is

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix},\, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

C34 (Robert Beezer) For the 3×4 matrix A and the column vector $\mathbf{y} \in \mathbb{C}^4$ given below, determine if $\mathbf{y}$ is in the row space of A. In other words, answer the question: $\mathbf{y} \in \mathcal{R}(A)$?

$$A = \begin{bmatrix} -2 & 6 & 7 & -1 \\ 7 & -3 & 0 & -3 \\ 8 & 0 & 7 & 6 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -2 \end{bmatrix}$$

Solution (Robert Beezer)

$$\begin{aligned} \mathbf{y} \in \mathcal{R}(A) &\iff \mathbf{y} \in \mathcal{C}\left(A^t\right) && \text{Definition RSM} \\ &\iff \mathcal{LS}\left(A^t,\, \mathbf{y}\right) \text{ is consistent} && \text{Theorem CSCS} \end{aligned}$$

The augmented matrix $[\,A^t \mid \mathbf{y}]$ row reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

and with a leading 1 in the final column Theorem RCLS tells us the linear system is inconsistent and so $\mathbf{y} \notin \mathcal{R}(A)$.

C35 (Robert Beezer) For the matrix A below, find two different linearly independent sets whose spans equal the column space of A, $\mathcal{C}(A)$, such that

1. the elements are each columns of A.
2. the set is obtained by a procedure that is substantially different from the procedure you use in part (1).

$$A = \begin{bmatrix} 3 & 5 & 1 & -2 \\ 1 & 2 & 3 & 3 \\ -3 & -4 & 7 & 13 \end{bmatrix}$$

Solution (Robert Beezer) (a) By Theorem BCS we can row-reduce A, identify pivot columns with the set D, and "keep" those columns of A and we will have a set with the desired properties.

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -13 & -19 \\ 0 & \boxed{1} & 8 & 11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So we have the set of pivot columns $D = \{1, 2\}$ and we "keep" the first two columns of A,

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -4 \end{bmatrix} \right\}$$

(b) We can view the column space as the row space of the transpose (Theorem CSRST). We can get a basis of the row space of a matrix quickly by bringing the matrix to reduced row-echelon form and keeping the nonzero rows as column vectors (Theorem BRS). Here goes.

$$A^t \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -2 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Taking the nonzero rows and tilting them up as columns gives us

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

An approach based on the matrix L from extended echelon form (Definition EEF) and Theorem FS will work as well as an alternative approach.

C40 (Robert Beezer) The following archetypes are systems of equations. For each system, write the vector of constants as a linear combination of the vectors in the span construction for the column space provided by Theorem BCS (these vectors are listed for each of these archetypes).

Archetype A, Archetype B, Archetype C, Archetype D, Archetype E, Archetype F, Archetype G, Archetype H, Archetype I, Archetype J

C42 (Robert Beezer) The following archetypes are either matrices or systems of equations with coefficient matrices. For each matrix, compute a set of column vectors such that (1) the vectors are columns of the matrix, (2) the set is linearly independent, and (3) the span of the set is the column space of the matrix. See Theorem BCS.

Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J, Archetype K, Archetype L

C50 (Robert Beezer) The following archetypes are either matrices or systems of equations with coefficient matrices. For each matrix, compute a set of column vectors such that (1) the set is linearly independent, and (2) the span of the set is the row space of the matrix. See Theorem BRS.

Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J, Archetype K, Archetype L

C51 (Robert Beezer) The following archetypes are either matrices or systems of equations with coefficient matrices. For each matrix, compute the column space as the span of a linearly independent set as follows:

transpose the matrix, row-reduce, toss out zero rows, convert rows into column vectors. See Example CSROI.

Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J, Archetype K, Archetype L

C52 (Robert Beezer) The following archetypes are systems of equations. For each different coefficient matrix build two new vectors of constants. The first should lead to a consistent system and the second should lead to an inconsistent system. Descriptions of the column space as spans of linearly independent sets of vectors with "nice patterns" of zeros and ones might be most useful and instructive in connection with this exercise. (See the end of Example CSROI.)

Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J

M10 (Robert Beezer) For the matrix E below, find vectors $\mathbf{b}$ and $\mathbf{c}$ so that the system $\mathcal{LS}(E,\,\mathbf{b})$ is consistent and $\mathcal{LS}(E,\,\mathbf{c})$ is inconsistent.

$$E = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 3 & -1 & 0 & 2 \\ 4 & 1 & 1 & 6 \end{bmatrix}$$

Solution (Robert Beezer) Any vector from $\mathbb{C}^3$ will lead to a consistent system, and therefore there is no vector that will lead to an inconsistent system.

How do we convince ourselves of this? First, row-reduce E,

$$E \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix}$$

If we augment E with any vector of constants, and row-reduce the augmented matrix, we will never find a leading 1 in the final column, so by Theorem RCLS the system will always be consistent.

Said another way, the column space of E is all of $\mathbb{C}^3$, $\mathcal{C}(E) = \mathbb{C}^3$. So by Theorem CSCS any vector of constants will create a consistent system (and none will create an inconsistent system).

M20 (Robert Beezer) Usually the column space and null space of a matrix contain vectors of different sizes. For a square matrix, though, the vectors in these two sets are the same size. Usually the two sets will be different. Construct an example of a square matrix where the column space and null space are equal.

Solution (Robert Beezer) The 2×2 matrix

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

has $\mathcal{C}(A) = \mathcal{N}(A) = \left\langle\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}\right\rangle$.

M21 (Robert Beezer) We have a variety of theorems about how to create column spaces and row spaces and they frequently involve row-reducing a matrix. Here is a procedure that some try to use to get a column space. Begin with an $m \times n$ matrix A and row-reduce to a matrix B with columns $\mathbf{B}_1,\,\mathbf{B}_2,\,\mathbf{B}_3,\,\ldots,\,\mathbf{B}_n$. Then form the column space of A as

$$\mathcal{C}(A) = \langle\{\mathbf{B}_1,\,\mathbf{B}_2,\,\mathbf{B}_3,\,\ldots,\,\mathbf{B}_n\}\rangle = \mathcal{C}(B)$$

This is *not* not a legitimate procedure, and therefore is *not* a theorem. Construct an example to show that the procedure will not in general create the column space of A.

Solution (Robert Beezer) Begin with a matrix A (of any size) that does not have any zero rows, but which when row-reduced to B yields at least one row of zeros. Such a matrix should be easy to construct (or find, like say from Archetype A).

$\mathcal{C}(A)$ will contain some vectors whose final slot (entry m) is non-zero, however, every column vector from the matrix B will have a zero in slot m and so every vector in $\mathcal{C}(B)$ will also contain a zero in the final slot. This means that $\mathcal{C}(A) \neq \mathcal{C}(B)$, since we have vectors in $\mathcal{C}(A)$ that cannot be elements of $\mathcal{C}(B)$.

T40 (Robert Beezer) Suppose that A is an $m \times n$ matrix and B is an $n \times p$ matrix. Prove that the column space of AB is a subset of the column space of A, that is $\mathcal{C}(AB) \subseteq \mathcal{C}(A)$. Provide an example where the opposite is false, in other words give an example where $\mathcal{C}(A) \not\subseteq \mathcal{C}(AB)$. (Compare with Exercise MM.T40.)

Solution (Robert Beezer) Choose $\mathbf{x} \in \mathcal{C}(AB)$. Then by Theorem CSCS there is a vector $\mathbf{w}$ that is a solution to $\mathcal{LS}(AB,\,\mathbf{x})$. Define the vector $\mathbf{y}$ by $\mathbf{y} = B\mathbf{w}$. We're set,

$$\begin{aligned} A\mathbf{y} &= A\left(B\mathbf{w}\right) && \text{Definition of } \mathbf{y} \\ &= \left(AB\right)\mathbf{w} && \text{Theorem MMA} \\ &= \mathbf{x} && \mathbf{w} \text{ solution to } \mathcal{LS}(AB,\,\mathbf{x}) \end{aligned}$$

This says that $\mathcal{LS}(A,\,\mathbf{x})$ is a consistent system, and by Theorem CSCS, we see that $\mathbf{x} \in \mathcal{C}(A)$ and therefore $\mathcal{C}(AB) \subseteq \mathcal{C}(A)$.

For an example where $\mathcal{C}(A) \not\subseteq \mathcal{C}(AB)$ choose A to be any nonzero matrix and choose B to be a zero matrix. Then $\mathcal{C}(A) \neq \{\mathbf{0}\}$ and $\mathcal{C}(AB) = \mathcal{C}(\mathcal{O}) = \{\mathbf{0}\}$.

T41 (Robert Beezer) Suppose that A is an $m \times n$ matrix and B is an $n \times n$ nonsingular matrix. Prove that the column space of A is equal to the column space of AB, that is $\mathcal{C}(A) = \mathcal{C}(AB)$. (Compare with Exercise MM.T41 and Exercise CRS.T40.)

Solution (Robert Beezer) From the solution to Exercise CRS.T40 we know that $\mathcal{C}(AB) \subseteq \mathcal{C}(A)$. So to establish the set equality (Definition SE) we need to show that $\mathcal{C}(A) \subseteq \mathcal{C}(AB)$.

Choose $\mathbf{x} \in \mathcal{C}(A)$. By Theorem CSCS the linear system $\mathcal{LS}(A,\,\mathbf{x})$ is consistent, so let $\mathbf{y}$ be one such solution. Because B is nonsingular, and linear system using B as a coefficient matrix will have a solution (Theorem NMUS). Let $\mathbf{w}$ be the unique solution to the linear system $\mathcal{LS}(B,\,\mathbf{y})$. All set, here we go,

$$\begin{aligned} \left(AB\right)\mathbf{w} &= A\left(B\mathbf{w}\right) && \text{Theorem MMA} \\ &= A\mathbf{y} && \mathbf{w} \text{ solution to } \mathcal{LS}(B,\,\mathbf{y}) \\ &= \mathbf{x} && \mathbf{y} \text{ solution to } \mathcal{LS}(A,\,\mathbf{x}) \end{aligned}$$

This says that the linear system $\mathcal{LS}(AB,\,\mathbf{x})$ is consistent, so by Theorem CSCS, $\mathbf{x} \in \mathcal{C}(AB)$. So $\mathcal{C}(A) \subseteq \mathcal{C}(AB)$.

T45 (Robert Beezer) Suppose that A is an $m \times n$ matrix and B is an $n \times m$ matrix where AB is a nonsingular matrix. Prove that

1. $\mathcal{N}(B) = \{\mathbf{0}\}$
2. $\mathcal{C}(B) \cap \mathcal{N}(A) = \{\mathbf{0}\}$

Discuss the case when $m = n$ in connection with Theorem NPNT.

Solution (Robert Beezer) First, $\mathbf{0} \in \mathcal{N}(B)$ trivially. Now suppose that $\mathbf{x} \in \mathcal{N}(B)$. Then

$$\begin{aligned} AB\mathbf{x} &= A(B\mathbf{x}) && \text{Theorem MMA} \\ &= A\mathbf{0} && \mathbf{x} \in \mathcal{N}(B) \\ &= \mathbf{0} && \text{Theorem MMZM} \end{aligned}$$

Since we have assumed AB is nonsingular, Definition NM implies that $\mathbf{x} = \mathbf{0}$.

Second, $\mathbf{0} \in \mathcal{C}(B)$ and $\mathbf{0} \in \mathcal{N}(A)$ trivially, and so the zero vector is in the intersection as well (Definition SI). Now suppose that $\mathbf{y} \in \mathcal{C}(B) \cap \mathcal{N}(A)$. Because $\mathbf{y} \in \mathcal{C}(B)$, Theorem CSCS says the system $\mathcal{LS}(B,\,\mathbf{y})$ is consistent. Let $\mathbf{x} \in \mathbb{C}^n$ be one solution to this system. Then

$$\begin{aligned} AB\mathbf{x} &= A(B\mathbf{x}) && \text{Theorem MMA} \\ &= A\mathbf{y} && \mathbf{x} \text{ solution to } \mathcal{LS}(B,\,\mathbf{y}) \end{aligned}$$

$$= \mathbf{0} \qquad \mathbf{y} \in \mathcal{N}(A)$$

Since we have assumed AB is nonsingular, Definition NM implies that $\mathbf{x} = \mathbf{0}$. Then $\mathbf{y} = B\mathbf{x} = B\mathbf{0} = \mathbf{0}$.

When AB is nonsingular and $m = n$ we know that the first condition, $\mathcal{N}(B) = \{\mathbf{0}\}$, means that B is nonsingular (Theorem NMTNS). Because B is nonsingular Theorem CSNM implies that $\mathcal{C}(B) = \mathbb{C}^m$. In order to have the second condition fulfilled, $\mathcal{C}(B) \cap \mathcal{N}(A) = \{\mathbf{0}\}$, we must realize that $\mathcal{N}(A) = \{\mathbf{0}\}$. However, a second application of Theorem NMTNS shows that A must be nonsingular. This reproduces Theorem NPNT.

Section FS
Four Subsets

C20 (Robert Beezer) Example FSAG concludes with several questions. Perform the analysis suggested by these questions.

C25 (Robert Beezer) Given the matrix A below, use the extended echelon form of A to answer each part of this problem. In each part, find a linearly independent set of vectors, S, so that the span of S, $\langle S \rangle$, equals the specified set of vectors.

$$A = \begin{bmatrix} -5 & 3 & -1 \\ -1 & 1 & 1 \\ -8 & 5 & -1 \\ 3 & -2 & 0 \end{bmatrix}$$

1. The row space of A, $\mathcal{R}(A)$.
2. The column space of A, $\mathcal{C}(A)$.
3. The null space of A, $\mathcal{N}(A)$.
4. The left null space of A, $\mathcal{L}(A)$.

Solution (Robert Beezer) Add a 4×4 identity matrix to the right of A to form the matrix M and then row-reduce to the matrix N,

$$M = \begin{bmatrix} -5 & 3 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ -8 & 5 & -1 & 0 & 0 & 1 & 0 \\ 3 & -2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2 & 0 & 0 & -2 & -5 \\ 0 & \boxed{1} & 3 & 0 & 0 & -3 & -8 \\ 0 & 0 & 0 & \boxed{1} & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 & 3 \end{bmatrix} = N$$

To apply Theorem FS in each of these four parts, we need the two matrices,

$$C = \begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & 3 \end{bmatrix} \qquad L = \begin{bmatrix} \boxed{1} & 0 & -1 & -1 \\ 0 & \boxed{1} & 1 & 3 \end{bmatrix}$$

(a)

$$\mathcal{R}(A) = \mathcal{R}(C) \qquad \text{Theorem FS}$$
$$= \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\} \right\rangle \qquad \text{Theorem BRS}$$

(b)

$$\mathcal{C}(A) = \mathcal{N}(L) \qquad \text{Theorem FS}$$
$$= \left\langle \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle \qquad \text{Theorem BNS}$$

(c)

$$\mathcal{N}(A) = \mathcal{N}(C) \qquad \text{Theorem FS}$$

$$= \left\langle \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} \right\rangle \qquad \text{Theorem BNS}$$

(d)

$$\mathcal{L}(A) = \mathcal{R}(L) \qquad \text{Theorem FS}$$

$$= \left\langle \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\rangle \qquad \text{Theorem BRS}$$

C26 (Robert Beezer) For the matrix D below use the extended echelon form to find:

1. A linearly independent set whose span is the column space of D.
2. A linearly independent set whose span is the left null space of D.

$$D = \begin{bmatrix} -7 & -11 & -19 & -15 \\ 6 & 10 & 18 & 14 \\ 3 & 5 & 9 & 7 \\ -1 & -2 & -4 & -3 \end{bmatrix}$$

Solution (Robert Beezer) For both parts, we need the extended echelon form of the matrix.

$$\begin{bmatrix} -7 & -11 & -19 & -15 & 1 & 0 & 0 & 0 \\ 6 & 10 & 18 & 14 & 0 & 1 & 0 & 0 \\ 3 & 5 & 9 & 7 & 0 & 0 & 1 & 0 \\ -1 & -2 & -4 & -3 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -2 & -1 & 0 & 0 & 2 & 5 \\ 0 & \boxed{1} & 3 & 2 & 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & -2 & 0 \end{bmatrix}$$

From this matrix we extract the last two rows, in the last four columns to form the matrix L,

$$L = \begin{bmatrix} \boxed{1} & 0 & 3 & 2 \\ 0 & \boxed{1} & -2 & 0 \end{bmatrix}$$

By Theorem FS and Theorem BNS we have

$$\mathcal{C}(D) = \mathcal{N}(L) = \left\langle \left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$$

By Theorem FS and Theorem BRS we have

$$\mathcal{L}(D) = \mathcal{R}(L) = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix} \right\} \right\rangle$$

C41 (Robert Beezer) The following archetypes are systems of equations. For each system, write the vector of constants as a linear combination of the vectors in the span construction for the column space provided by Theorem FS and Theorem BNS (these vectors are listed for each of these archetypes).

Archetype A, Archetype B, Archetype C, Archetype D, Archetype E, Archetype F, Archetype G, Archetype H, Archetype I, Archetype J

C43 (Robert Beezer) The following archetypes are either matrices or systems of equations with coefficient matrices. For each matrix, compute the extended echelon form N and identify the matrices C and L. Using Theorem FS, Theorem BNS and Theorem BRS express the null space, the row space, the column space and left null space of each coefficient matrix as a span of a linearly independent set.

Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J, Archetype K, Archetype L

C60 (Robert Beezer) For the matrix B below, find sets of vectors whose span equals the column space of B ($\mathcal{C}(B)$) and which individually meet the following extra requirements.

1. The set illustrates the definition of the column space.
2. The set is linearly independent and the members of the set are columns of B.
3. The set is linearly independent with a "nice pattern of zeros and ones" at the *top* of each vector.
4. The set is linearly independent with a "nice pattern of zeros and ones" at the *bottom* of each vector.

$$B = \begin{bmatrix} 2 & 3 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & 2 & 3 & -4 \end{bmatrix}$$

Solution (Robert Beezer) The definition of the column space is the span of the set of columns (Definition CSM). So the desired set is just the four columns of B,

$$S = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} \right\}$$

Theorem BCS suggests row-reducing the matrix and using the columns of B that correspond to the pivot columns.

$$B \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 & 2 \\ 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the pivot columns are numbered by elements of $D = \{1,\,2\}$, so the requested set is

$$S = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\}$$

We can find this set by row-reducing the transpose of B, deleting the zero rows, and using the nonzero rows as column vectors in the set. This is an application of Theorem CSRST followed by Theorem BRS.

$$B^t \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 3 \\ 0 & \boxed{1} & -7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So the requested set is

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -7 \end{bmatrix} \right\}$$

With the column space expressed as a null space, the vectors obtained via Theorem BNS will be of the desired shape. So we first proceed with Theorem FS and create the extended echelon form,

$$[B \mid I_3] = \begin{bmatrix} 2 & 3 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 2 & 3 & -4 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 & 2 & 0 & \frac{2}{3} & \frac{-1}{3} \\ 0 & \boxed{1} & 1 & -1 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \boxed{1} & \frac{-7}{3} & \frac{-1}{3} \end{bmatrix}$$

So, employing Theorem FS, we have $\mathcal{C}(B) = \mathcal{N}(L)$, where

$$L = \begin{bmatrix} \boxed{1} & \frac{-7}{3} & \frac{-1}{3} \end{bmatrix}$$

We can find the desired set of vectors from Theorem BNS as

$$S = \left\{ \begin{bmatrix} \frac{7}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix} \right\}$$

C61 (Robert Beezer) Let A be the matrix below, and find the indicated sets with the requested properties.

$$A = \begin{bmatrix} 2 & -1 & 5 & -3 \\ -5 & 3 & -12 & 7 \\ 1 & 1 & 4 & -3 \end{bmatrix}$$

1. A linearly independent set S so that $\mathcal{C}(A) = \langle S \rangle$ and S is composed of columns of A.

2. A linearly independent set S so that $\mathcal{C}(A) = \langle S \rangle$ and the vectors in S have a nice pattern of zeros and ones at the top of the vectors.

3. A linearly independent set S so that $\mathcal{C}(A) = \langle S \rangle$ and the vectors in S have a nice pattern of zeros and ones at the bottom of the vectors.

4. A linearly independent set S so that $\mathcal{R}(A) = \langle S \rangle$.

Solution (Robert Beezer) First find a matrix B that is row-equivalent to A and in reduced row-echelon form

$$B = \begin{bmatrix} \boxed{1} & 0 & 3 & -2 \\ 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem BCS we can choose the columns of A that correspond to dependent variables ($D = \{1, 2\}$) as the elements of S and obtain the desired properties. So

$$S = \left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\}$$

We can write the column space of A as the row space of the transpose (Theorem CSRST). So we row-reduce the transpose of A to obtain the row-equivalent matrix C in reduced row-echelon form

$$C = \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The nonzero rows (written as columns) will be a linearly independent set that spans the row space of A^t, by Theorem BRS, and the zeros and ones will be at the top of the vectors,

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

In preparation for Theorem FS, augment A with the 3×3 identity matrix I_3 and row-reduce to obtain the extended echelon form,

$$\begin{bmatrix} 1 & 0 & 3 & -2 & 0 & -\frac{1}{8} & \frac{3}{8} \\ 0 & 1 & 1 & -1 & 0 & \frac{1}{8} & \frac{5}{8} \\ 0 & 0 & 0 & 0 & 1 & \frac{3}{8} & -\frac{1}{8} \end{bmatrix}$$

Then since the first four columns of row 3 are all zeros, we extract

$$L = \begin{bmatrix} \boxed{1} & \frac{3}{8} & -\frac{1}{8} \end{bmatrix}$$

Theorem FS says that $\mathcal{C}(A) = \mathcal{N}(L)$. We can then use Theorem BNS to construct the desired set S, based on the free variables with indices in $F = \{2, 3\}$ for the homogeneous system $\mathcal{LS}(L, \mathbf{0})$, so

$$S = \left\{ \begin{bmatrix} -\frac{3}{8} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{8} \\ 0 \\ 1 \end{bmatrix} \right\}$$

Notice that the zeros and ones are at the bottom of the vectors.
This is a straightforward application of Theorem BRS. Use the row-reduced matrix B from part (a), grab the nonzero rows, and write them as column vectors,

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

M50 (Robert Beezer) Suppose that A is a nonsingular matrix. Extend the four conclusions of Theorem FS in this special case and discuss connections with previous results (such as Theorem NME4).

M51 (Robert Beezer) Suppose that A is a singular matrix. Extend the four conclusions of Theorem FS in this special case and discuss connections with previous results (such as Theorem NME4).

Chapter VS
Vector Spaces

Section VS
Vector Spaces

M10 (Robert Beezer) Define a possibly new vector space by beginning with the set and vector addition from $\mathbb{C}^2$ (Example VSCV) but change the definition of scalar multiplication to

$$\alpha\mathbf{x} = \mathbf{0} = \begin{bmatrix}0\\0\end{bmatrix} \qquad \alpha \in \mathbb{C},\ \mathbf{x} \in \mathbb{C}^2$$

Prove that the first nine properties required for a vector space hold, but Property O does not hold.

This example shows us that we cannot expect to be able to derive Property O as a consequence of assuming the first nine properties. In other words, we cannot slim down our list of properties by jettisoning the last one, and still have the same collection of objects qualify as vector spaces.

M11 (Chris Black) Let V be the set $\mathbb{C}^2$ with the usual vector addition, but with scalar multiplication defined by

$$\alpha\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}\alpha y\\ \alpha x\end{bmatrix}$$

Determine whether or not V is a vector space with these operations.

Solution (Chris Black) The set $\mathbb{C}^2$ with the proposed operations is not a vector space since Property O is not valid. A counterexample is $1\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix} \neq \begin{bmatrix}3\\2\end{bmatrix}$, so in general, $1\mathbf{u} \neq \mathbf{u}$.

M12 (Chris Black) Let V be the set $\mathbb{C}^2$ with the usual scalar multiplication, but with vector addition defined by

$$\begin{bmatrix}x\\y\end{bmatrix} + \begin{bmatrix}z\\w\end{bmatrix} = \begin{bmatrix}y+w\\x+z\end{bmatrix}$$

Determine whether or not V is a vector space with these operations.

Solution (Chris Black) Let's consider the existence of a zero vector, as required by Property Z of a vector space. The "regular" zero vector fails : $\begin{bmatrix}x\\y\end{bmatrix} + \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}y\\x\end{bmatrix} \neq \begin{bmatrix}x\\y\end{bmatrix}$ (remember that the property must hold for *every*must be a non-empty set, be we vector, not just for some). Is there another vector that fills the role of the zero vector? Suppose that $\mathbf{0} = \begin{bmatrix}z_1\\z_2\end{bmatrix}$. Then for any vector $\begin{bmatrix}x\\y\end{bmatrix}$, we have

$$\begin{bmatrix}x\\y\end{bmatrix} + \begin{bmatrix}z_1\\z_2\end{bmatrix} = \begin{bmatrix}y+z_2\\x+z_1\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$$

so that $x = y + z_2$ and $y = x + z_1$. This means that $z_1 = y - x$ and $z_2 = x - y$. However, since x and y can be any complex numbers, there are no fixed complex numbers z_1 and z_2 that satisfy these equations. Thus, there is no zero vector, Property Z is not valid, and the set $\mathbb{C}^2$ with the proposed operations is not a vector space.

M13 (Chris Black) Let V be the set $M_{2,2}$ with the usual scalar multiplication, but with addition defined by $A + B = \mathcal{O}_{2,2}$ for all 2×2 matrices A and B. Determine whether or not V is a vector space with these operations.

Solution (Chris Black) Since scalar multiplication remains unchanged, we only need to consider the axioms that involve vector addition. Since every sum is the zero matrix, the first 4 properties hold easily. However, there is no zero vector in this set. Suppose that there was. Then there is a matrix Z so that $A + Z = A$ for any 2×2 matrix A. However, $A + Z = \mathcal{O}_{2,2}$, which is in general not equal to A, so Property Z fails and this set is not a vector space.

M14 (Chris Black) Let V be the set $M_{2,2}$ with the usual addition, but with scalar multiplication defined by $\alpha A = \mathcal{O}_{2,2}$ for all 2×2 matrices A and scalars α. Determine whether or not V is a vector space with these operations.

Solution (Chris Black) Since addition is unchanged, we only need to check the axioms involving scalar multiplication. The proposed scalar multiplication clearly fails Property O : $1A = \mathcal{O}_{2,2} \neq A$. Thus, the proposed set is not a vector space.

M15 (Chris Black) Consider the following sets of 3×3 matrices, where the symbol $*$ indicates the position of an arbitrary complex number. Determine whether or not these sets form vector spaces with the usual operations of addition and scalar multiplication for matrices.

1. All matrices of the form $\begin{bmatrix} * & * & 1 \\ * & 1 & * \\ 1 & * & * \end{bmatrix}$
2. All matrices of the form $\begin{bmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{bmatrix}$
3. All matrices of the form $\begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$ (These are the **diagonal** matrices.)
4. All matrices of the form $\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$ (These are the **upper triangular** matrices.)

Solution (Chris Black) There is something to notice here that will make our job much easier: Since each of these sets are comprised of 3×3 matrices with the standard operations of addition and scalar multiplication of matrices, the last 8 properties will automatically hold. That is, we really only need to verify Property AC and Property SC.

- This set is not closed under either scalar multiplication or addition (fails Property AC and Property SC). For example, $3 \begin{bmatrix} * & * & 1 \\ * & 1 & * \\ 1 & * & * \end{bmatrix} = \begin{bmatrix} * & * & 3 \\ * & 3 & * \\ 3 & * & * \end{bmatrix}$ is not a member of the proposed set.
- This set is closed under both scalar multiplication and addition, so this set is a vector space with the standard operation of addition and scalar multiplication.
- This set is closed under both scalar multiplication and addition, so this set is a vector space with the standard operation of addition and scalar multiplication.
- This set is closed under both scalar multiplication and addition, so this set is a vector space with the standard operation of addition and scalar multiplication.

M20 (Chris Black) Explain why we need to define the vector space P_n as the set of all polynomials with degree *up to and including* n instead of the more obvious set of all polynomials of degree *exactly* n.

Solution (Chris Black) Hint: The set of all polynomials of degree *exactly* n fails one of the closure properties of a vector space. Which one, and why?

M21 (Chris Black) The set of integers is denoted $\mathbb{Z}$. Does the set $\mathbb{Z}^2 = \left\{\begin{bmatrix} m \\ n \end{bmatrix} \middle| m, n \in \mathbb{Z}\right\}$ with the operations of standard addition and scalar multiplication of vectors form a vector space?

Solution (Robert Beezer) Additive closure will hold, but scalar closure will not. The best way to convince yourself of this is to construct a counterexample. Such as, $\frac{1}{2} \in \mathbb{C}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{Z}^2$, however $\frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \notin \mathbb{Z}^2$, which violates Property SC. So $\mathbb{Z}^2$ is not a vector space.

T10 (Robert Beezer) Prove each of the ten properties of Definition VS for each of the following examples of a vector space:
Example VSP
Example VSIS
Example VSF
Example VSS

The next three problems suggest that under the right situations we can "cancel." In practice, these techniques should be avoided in other proofs. Prove each of the following statements.

T21 (Robert Beezer) Suppose that V is a vector space, and $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$. If $\mathbf{w} + \mathbf{u} = \mathbf{w} + \mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.

Solution (Robert Beezer)

$$
\begin{aligned}
\mathbf{u} &= \mathbf{0} + \mathbf{u} && \text{Property Z} \\
&= (-\mathbf{w} + \mathbf{w}) + \mathbf{u} && \text{Property AI} \\
&= -\mathbf{w} + (\mathbf{w} + \mathbf{u}) && \text{Property AA} \\
&= -\mathbf{w} + (\mathbf{w} + \mathbf{v}) && \text{Hypothesis} \\
&= (-\mathbf{w} + \mathbf{w}) + \mathbf{v} && \text{Property AA} \\
&= \mathbf{0} + \mathbf{v} && \text{Property AI} \\
&= \mathbf{v} && \text{Property Z}
\end{aligned}
$$

T22 (Robert Beezer) Suppose V is a vector space, $\mathbf{u}$, $\mathbf{v} \in V$ and α is a nonzero scalar from $\mathbb{C}$. If $\alpha\mathbf{u} = \alpha\mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.

Solution (Robert Beezer)

$$
\begin{aligned}
\mathbf{u} &= 1\mathbf{u} && \text{Property O} \\
&= \left(\frac{1}{\alpha}\alpha\right)\mathbf{u} && \alpha \neq 0 \\
&= \frac{1}{\alpha}(\alpha\mathbf{u}) && \text{Property SMA} \\
&= \frac{1}{\alpha}(\alpha\mathbf{v}) && \text{Hypothesis} \\
&= \left(\frac{1}{\alpha}\alpha\right)\mathbf{v} && \text{Property SMA} \\
&= 1\mathbf{v} \\
&= \mathbf{v} && \text{Property O}
\end{aligned}
$$

T23 (Robert Beezer) Suppose V is a vector space, $\mathbf{u} \neq \mathbf{0}$ is a vector in V and α, $\beta \in \mathbb{C}$. If $\alpha\mathbf{u} = \beta\mathbf{u}$, then $\alpha = \beta$.

Solution (Robert Beezer)

$$
\begin{aligned}
\mathbf{0} &= \alpha\mathbf{u} + -(\alpha\mathbf{u}) && \text{Property AI} \\
&= \beta\mathbf{u} + -(\alpha\mathbf{u}) && \text{Hypothesis} \\
&= \beta\mathbf{u} + (-1)(\alpha\mathbf{u}) && \text{Theorem AISM} \\
&= \beta\mathbf{u} + ((-1)\alpha)\mathbf{u} && \text{Property SMA}
\end{aligned}
$$

$$\begin{aligned} &= \beta\mathbf{u} + (-\alpha)\,\mathbf{u} \\ &= (\beta - \alpha)\,\mathbf{u} && \text{Property DSA} \end{aligned}$$

By hypothesis, $\mathbf{u} \neq \mathbf{0}$, so Theorem SMEZV implies

$$\begin{aligned} 0 &= \beta - \alpha \\ \alpha &= \beta \end{aligned}$$

T30 (Robert Beezer) Suppose that V is a vector space and $\alpha \in \mathbb{C}$ is a scalar such that $\alpha\mathbf{x} = \mathbf{x}$ for every $\mathbf{x} \in V$. Prove that $\alpha = 1$. In other words, Property O is not duplicated for any other scalar but the "special" scalar, 1. (This question was suggested by James Gallagher.)

Solution (Robert Beezer) We have,

$$\begin{aligned} \mathbf{0} &= \mathbf{x} - \mathbf{x} && \text{Property AI} \\ &= \alpha\mathbf{x} - \mathbf{x} && \text{Hypothesis} \\ &= \alpha\mathbf{x} - 1\mathbf{x} && \text{Property O} \\ &= (\alpha - 1)\mathbf{x} && \text{Property DSA} \end{aligned}$$

So by Theorem SMEZV we conclude that $\alpha - 1 = 0$ or $\mathbf{x} = \mathbf{0}$. However, since our hypothesis was *for every* $\mathbf{x} \in V$, we are left with the first possibility and $\alpha = 1$.

There is one flaw in the proof above, and as stated, the problem is not correct either. Can you spot the flaw and as a result correct the problem statement? (Hint: Example VSS).

Section S
Subspaces

C15 (Chris Black) Working within the vector space $\mathbb{C}^3$, determine if $\mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$ is in the subspace W,

$$W = \left\langle \left\{ \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\} \right\rangle$$

Solution (Chris Black) For $\mathbf{b}$ to be an element of $W = \langle S \rangle$ there must be linear combination of the vectors in S that equals $\mathbf{b}$ (Definition SSCV). The existence of such scalars is equivalent to the linear system $\mathcal{LS}(A, \mathbf{b})$ being consistent, where A is the matrix whose columns are the vectors from S (Theorem SLSLC).

$$\begin{bmatrix} 3 & 1 & 1 & 2 & 4 \\ 2 & 0 & 1 & 1 & 3 \\ 3 & 3 & 0 & 3 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1/2 & 1/2 & 0 \\ 0 & \boxed{1} & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

So by Theorem RCLS the system is inconsistent, which indicates that $\mathbf{b}$ is not an element of the subspace W.

C16 (Chris Black) Working within the vector space $\mathbb{C}^4$, determine if $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ is in the subspace W,

$$W = \left\langle \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\} \right\rangle$$

Solution (Chris Black) For $\mathbf{b}$ to be an element of $W = \langle S \rangle$ there must be linear combination of the vectors in S that equals $\mathbf{b}$ (Definition SSCV). The existence of such scalars is equivalent to the linear system

$\mathcal{LS}(A,\,\mathbf{b})$ being consistent, where A is the matrix whose columns are the vectors from S (Theorem SLSLC).

$$\begin{bmatrix}1 & 1 & 2 & 1\\ 2 & 0 & 1 & 1\\ -1 & 3 & 1 & 0\\ 1 & 1 & 2 & 1\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & 0 & 1/3\\ 0 & \boxed{1} & 0 & 0\\ 0 & 0 & \boxed{1} & 1/3\\ 0 & 0 & 0 & 0\end{bmatrix}$$

So by Theorem RCLS the system is consistent, which indicates that $\mathbf{b}$ is in the subspace W.

C17 (Chris Black) Working within the vector space $\mathbb{C}^4$, determine if $\mathbf{b} = \begin{bmatrix}2\\1\\2\\1\end{bmatrix}$ is in the subspace W,

$$W = \left\langle \left\{ \begin{bmatrix}1\\2\\0\\2\end{bmatrix}, \begin{bmatrix}1\\0\\3\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\1\\2\\0\end{bmatrix} \right\} \right\rangle$$

Solution (Chris Black) For $\mathbf{b}$ to be an element of $W = \langle S\rangle$ there must be linear combination of the vectors in S that equals $\mathbf{b}$ (Definition SSCV). The existence of such scalars is equivalent to the linear system $\mathcal{LS}(A,\,\mathbf{b})$ being consistent, where A is the matrix whose columns are the vectors from S (Theorem SLSLC).

$$\begin{bmatrix}1 & 1 & 0 & 1 & 2\\ 2 & 0 & 1 & 1 & 1\\ 0 & 3 & 0 & 2 & 2\\ 2 & 1 & 2 & 0 & 0\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & 0 & 0 & 3/2\\ 0 & \boxed{1} & 0 & 0 & 1\\ 0 & 0 & \boxed{1} & 0 & -3/2\\ 0 & 0 & 0 & \boxed{1} & -1/2\end{bmatrix}$$

So by Theorem RCLS the system is consistent, which indicates that $\mathbf{b}$ is in the subspace W.

C20 (Robert Beezer) Working within the vector space P_3 of polynomials of degree 3 or less, determine if $p(x) = x^3 + 6x + 4$ is in the subspace W below.

$$W = \left\langle \left\{ x^3 + x^2 + x,\, x^3 + 2x - 6,\, x^2 - 5 \right\} \right\rangle$$

Solution (Robert Beezer) The question is if p can be written as a linear combination of the vectors in W. To check this, we set p equal to a linear combination and massage with the definitions of vector addition and scalar multiplication that we get with P_3 (Example VSP)

$$\begin{aligned} p(x) &= a_1(x^3 + x^2 + x) + a_2(x^3 + 2x - 6) + a_3(x^2 - 5)\\ x^3 + 6x + 4 &= (a_1 + a_2)x^3 + (a_1 + a_3)x^2 + (a_1 + 2a_2)x + (-6a_2 - 5a_3) \end{aligned}$$

Equating coefficients of equal powers of x, we get the system of equations,

$$\begin{aligned} a_1 + a_2 &= 1\\ a_1 + a_3 &= 0\\ a_1 + 2a_2 &= 6\\ -6a_2 - 5a_3 &= 4 \end{aligned}$$

The augmented matrix of this system of equations row-reduces to

$$\begin{bmatrix}\boxed{1} & 0 & 0 & 0\\ 0 & \boxed{1} & 0 & 0\\ 0 & 0 & \boxed{1} & 0\\ 0 & 0 & 0 & \boxed{1}\end{bmatrix}$$

There is a leading 1 in the last column, so Theorem RCLS implies that the system is inconsistent. So there is no way for p to gain membership in W, so $p \notin W$.

C21 (Robert Beezer) Consider the subspace

$$W = \left\langle \left\{ \begin{bmatrix}2 & 1\\ 3 & -1\end{bmatrix}, \begin{bmatrix}4 & 0\\ 2 & 3\end{bmatrix}, \begin{bmatrix}-3 & 1\\ 2 & 1\end{bmatrix} \right\} \right\rangle$$

of the vector space of 2×2 matrices, M_{22}. Is $C = \begin{bmatrix} -3 & 3 \\ 6 & -4 \end{bmatrix}$ an element of W?

Solution (Robert Beezer) In order to belong to W, we must be able to express C as a linear combination of the elements in the spanning set of W. So we begin with such an expression, using the unknowns a, b, c for the scalars in the linear combination.

$$C = \begin{bmatrix} -3 & 3 \\ 6 & -4 \end{bmatrix} = a\begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} + b\begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix} + c\begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}$$

Massaging the right-hand side, according to the definition of the vector space operations in M_{22} (Example VSM), we find the matrix equality,

$$\begin{bmatrix} -3 & 3 \\ 6 & -4 \end{bmatrix} = \begin{bmatrix} 2a + 4b - 3c & a + c \\ 3a + 2b + 2c & -a + 3b + c \end{bmatrix}$$

Matrix equality allows us to form a system of four equations in three variables, whose augmented matrix row-reduces as follows,

$$\begin{bmatrix} 2 & 4 & -3 & -3 \\ 1 & 0 & 1 & 3 \\ 3 & 2 & 2 & 6 \\ -1 & 3 & 1 & -4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this system of equations is consistent (Theorem RCLS), a solution will provide values for a, b and c that allow us to recognize C as an element of W.

C25 (Robert Beezer) Show that the set $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \, 3x_1 - 5x_2 = 12 \right\}$ from Example NSC2Z fails Property AC and Property SC.

C26 (Robert Beezer) Show that the set $Y = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \, x_1 \in \mathbb{Z}, \, x_2 \in \mathbb{Z} \right\}$ from Example NSC2S has Property AC.

M20 (Robert Beezer) In $\mathbb{C}^3$, the vector space of column vectors of size 3, prove that the set Z is a subspace.

$$Z = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \, 4x_1 - x_2 + 5x_3 = 0 \right\}$$

Solution (Robert Beezer) The membership criteria for Z is a single linear equation, which comprises a homogeneous system of equations. As such, we can recognize Z as the solutions to this system, and therefore Z is a null space. Specifically, $Z = \mathcal{N}([4 \quad -1 \quad 5])$. Every null space is a subspace by Theorem NSMS.

A less direct solution appeals to Theorem TSS.

First, we want to be certain Z is non-empty. The zero vector of $\mathbb{C}^3$, $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, is a good candidate, since if it fails to be in Z, we will know that Z is *not* a vector space. Check that

$$4(0) - (0) + 5(0) = 0$$

so that $\mathbf{0} \in Z$.

Suppose $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ are vectors from Z. Then we know that these vectors cannot be totally arbitrary, they must have gained membership in Z by virtue of meeting the membership test. For example, we know that $\mathbf{x}$ must satisfy $4x_1 - x_2 + 5x_3 = 0$ while $\mathbf{y}$ must satisfy $4y_1 - y_2 + 5y_3 = 0$. Our second criteria asks the question, is $\mathbf{x} + \mathbf{y} \in Z$? Notice first that

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$

and we can test this vector for membership in Z as follows,

$$\begin{aligned}
&4(x_1 + y_1) - 1(x_2 + y_2) + 5(x_3 + y_3) \\
&= 4x_1 + 4y_1 - x_2 - y_2 + 5x_3 + 5y_3 \\
&= (4x_1 - x_2 + 5x_3) + (4y_1 - y_2 + 5y_3) \\
&= 0 + 0 && \mathbf{x} \in Z,\ \mathbf{y} \in Z \\
&= 0
\end{aligned}$$

and by this computation we see that $\mathbf{x} + \mathbf{y} \in Z$.

If $\alpha \in \mathbb{C}$ is a scalar and $\mathbf{x} \in Z$, is it always true that $\alpha\mathbf{x} \in Z$? To check our third criteria, we examine

$$\alpha\mathbf{x} = \alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix}$$

and we can test this vector for membership in Z with

$$\begin{aligned}
&4(\alpha x_1) - (\alpha x_2) + 5(\alpha x_3) \\
&\quad = \alpha(4x_1 - x_2 + 5x_3) \\
&\quad = \alpha 0 && \mathbf{x} \in Z \\
&\quad = 0
\end{aligned}$$

and we see that indeed $\alpha\mathbf{x} \in Z$. With the three conditions of Theorem TSS fulfilled, we can conclude that Z is a subspace of $\mathbb{C}^3$.

T20 (Robert Beezer) A square matrix A of size n is upper triangular if $[A]_{ij} = 0$ whenever $i > j$. Let UT_n be the set of all upper triangular matrices of size n. Prove that UT_n is a subspace of the vector space of all square matrices of size n, M_{nn}.

Solution (Robert Beezer) Apply Theorem TSS.

First, the zero vector of M_{nn} is the zero matrix, $\mathcal{O}$, whose entries are all zero (Definition ZM). This matrix then meets the condition that $[\mathcal{O}]_{ij} = 0$ for $i > j$ and so is an element of UT_n.

Suppose $A, B \in UT_n$. Is $A + B \in UT_n$? We examine the entries of $A + B$ "below" the diagonal. That is, in the following, assume that $i > j$.

$$\begin{aligned}
[A + B]_{ij} &= [A]_{ij} + [B]_{ij} && \text{Definition MA} \\
&= 0 + 0 && A, B \in UT_n \\
&= 0
\end{aligned}$$

which qualifies $A + B$ for membership in UT_n.

Suppose $\alpha \in \mathbb{C}$ and $A \in UT_n$. Is $\alpha A \in UT_n$? We examine the entries of αA "below" the diagonal. That is, in the following, assume that $i > j$.

$$\begin{aligned}
[\alpha A]_{ij} &= \alpha\, [A]_{ij} && \text{Definition MSM} \\
&= \alpha 0 && A \in UT_n \\
&= 0
\end{aligned}$$

which qualifies αA for membership in UT_n.

Having fulfilled the three conditions of Theorem TSS we see that UT_n is a subspace of M_{nn}.

T30 (Chris Black) Let P be the set of all polynomials, of any degree. The set P is a vector space. Let E be the subset of P consisting of all polynomials with only terms of even degree. Prove or disprove: the set E is a subspace of P.

Solution (Chris Black) **Proof:** Let E be the subset of P comprised of all polynomials with all terms of even degree. Clearly the set E is non-empty, as $z(x) = 0$ is a polynomial of even degree. Let $p(x)$ and $q(x)$

be arbitrary elements of E. Then there exist nonnegative integers m and n so that

$$\begin{aligned} p(x) &= a_0 + a_2x^2 + a_4x^4 + \cdots + a_{2n}x^{2n} \\ q(x) &= b_0 + b_2x^2 + b_4x^4 + \cdots + b_{2m}x^{2m} \end{aligned}$$

for some constants $a_0, a_2, \ldots, a_{2n}$ and $b_0, b_2, \ldots, b_{2m}$. Without loss of generality, we can assume that $m \leq n$. Thus, we have

$$p(x) + q(x) = (a_0 + b_0) + (a_2 + b_2)x^2 + \cdots + (a_{2m} + b_{2m})x^{2m} + a_{2m+2}x^{2m+2} + \cdots + a_{2n}x^{2n}$$

so $p(x) + q(x)$ has all even terms, and thus $p(x) + q(x) \in E$. Similarly, let α be a scalar. Then

$$\begin{aligned} \alpha p(x) &= \alpha(a_0 + a_2x^2 + a_4x^4 + \cdots + a_{2n}x^{2n}) \\ &= \alpha a_0 + (\alpha a_2)x^2 + (\alpha a_4)x^4 + \cdots + (\alpha a_{2n})x^{2n} \end{aligned}$$

so that $\alpha p(x)$ also has only terms of even degree, and $\alpha p(x) \in E$. Thus, E is a subspace of P.

T31 (Chris Black) Let P be the set of all polynomials, of any degree. The set P is a vector space. Let F be the subset of P consisting of all polynomials with only terms of odd degree. Prove or disprove: the set F is a subspace of P.

Solution (Chris Black) This conjecture is false. We know that the zero vector in P is the polynomial $z(x) = 0$, which does not have odd degree. Thus, the set F does not contain the zero vector, and cannot be a vector space.

Section LISS
Linear Independence and Spanning Sets

C20 (Robert Beezer) In the vector space of 2×2 matrices, M_{22}, determine if the set S below is linearly independent.

$$S = \left\{ \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \right\}$$

Solution (Robert Beezer) Begin with a relation of linear dependence on the vectors in S and massage it according to the definitions of vector addition and scalar multiplication in M_{22},

$$\begin{aligned} \mathcal{O} &= a_1 \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix} + a_3 \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 2a_1 + 4a_3 & -a_1 + 4a_2 + 2a_3 \\ a_1 - a_2 + a_3 & 3a_1 + 2a_2 + 3a_3 \end{bmatrix} \end{aligned}$$

By our definition of matrix equality (Definition ME) we arrive at a homogeneous system of linear equations,

$$\begin{aligned} 2a_1 + 4a_3 &= 0 \\ -a_1 + 4a_2 + 2a_3 &= 0 \\ a_1 - a_2 + a_3 &= 0 \\ 3a_1 + 2a_2 + 3a_3 &= 0 \end{aligned}$$

The coefficient matrix of this system row-reduces to the matrix,

$$\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$$

and from this we conclude that the only solution is $a_1 = a_2 = a_3 = 0$. Since the relation of linear dependence (Definition RLD) is trivial, the set S is linearly independent (Definition LI).

C21 (Robert Beezer) In the crazy vector space C (Example CVS), is the set $S = \{(0,\ 2),\ (2,\ 8)\}$ linearly independent?

Solution (Robert Beezer) We begin with a relation of linear dependence using unknown scalars a and b. We wish to know if these scalars *must* both be zero. Recall that the zero vector in C is $(-1,\ -1)$ and that the definitions of vector addition and scalar multiplication are not what we might expect.

$$\begin{aligned}
\mathbf{0} &= (-1,\,-1) \\
&= a(0,\,2) + b(2,\,8) && \text{Definition RLD} \\
&= (0a + a - 1,\,2a + a - 1) + (2b + b - 1,\,8b + b - 1) && \text{Scalar mult., Example CVS} \\
&= (a - 1,\,3a - 1) + (3b - 1,\,9b - 1) \\
&= (a - 1 + 3b - 1 + 1,\,3a - 1 + 9b - 1 + 1) && \text{Vector addition, Example CVS} \\
&= (a + 3b - 1,\,3a + 9b - 1)
\end{aligned}$$

From this we obtain two equalities, which can be converted to a homogeneous system of equations,

$$\begin{aligned} -1 &= a + 3b - 1 & a + 3b &= 0 \\ -1 &= 3a + 9b - 1 & 3a + 9b &= 0 \end{aligned}$$

This homogeneous system has a singular coefficient matrix, and so has more than just the trivial solution (Definition NM). Any nontrivial solution will give us a nontrivial relation of linear dependence on S. So S is linearly dependent (Definition LI).

C22 (Robert Beezer) In the vector space of polynomials P_3, determine if the set S is linearly independent or linearly dependent.

$$S = \left\{2 + x - 3x^2 - 8x^3,\, 1 + x + x^2 + 5x^3,\, 3 - 4x^2 - 7x^3\right\}$$

Solution (Robert Beezer) Begin with a relation of linear dependence (Definition RLD),

$$a_1\left(2 + x - 3x^2 - 8x^3\right) + a_2\left(1 + x + x^2 + 5x^3\right) + a_3\left(3 - 4x^2 - 7x^3\right) = \mathbf{0}$$

Massage according to the definitions of scalar multiplication and vector addition in the definition of P_3 (Example VSP) and use the zero vector for this vector space,

$$(2a_1 + a_2 + 3a_3) + (a_1 + a_2)\,x + (-3a_1 + a_2 - 4a_3)\,x^2 + (-8a_1 + 5a_2 - 7a_3)\,x^3 = 0 + 0x + 0x^2 + 0x^3$$

The definition of the equality of polynomials allows us to deduce the following four equations,

$$\begin{aligned} 2a_1 + a_2 + 3a_3 &= 0 \\ a_1 + a_2 &= 0 \\ -3a_1 + a_2 - 4a_3 &= 0 \\ -8a_1 + 5a_2 - 7a_3 &= 0 \end{aligned}$$

Row-reducing the coefficient matrix of this homogeneous system leads to the unique solution $a_1 = a_2 = a_3 = 0$. So the only relation of linear dependence on S is the trivial one, and this is linear independence for S (Definition LI).

C23 (Robert Beezer) Determine if the set $S = \{(3,\,1),\,(7,\,3)\}$ is linearly independent in the crazy vector space C (Example CVS).

Solution (Robert Beezer) Notice, or discover, that the following gives a nontrivial relation of linear dependence on S in C, so by Definition LI, the set S is linearly dependent.

$$2(3,\,1) + (-1)(7,\,3) = (7,\,3) + (-9,\,-5) = (-1,\,-1) = \mathbf{0}$$

C24 (Chris Black) In the vector space of real-valued functions $F = \{\,f \mid f : \mathbb{R} \to \mathbb{R}\}$, determine if the following set S is linearly independent.

$$S = \left\{\sin^2 x, \cos^2 x, 2\right\}$$

Solution (Chris Black) One of the fundamental identities of trigonometry is $\sin^2(x) + \cos^2(x) = 1$. Thus, we have a dependence relation $2(\sin^2 x) + 2(\cos^2 x) + (-1)(2) = 0$, and the set is linearly dependent.

C25 (Chris Black) Let

$$S = \left\{\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}, \begin{bmatrix}2 & 1\\-1 & 2\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 2\end{bmatrix}\right\}$$

1. Determine if S spans $M_{2,2}$.

2. Determine if S is linearly independent.

Solution (Chris Black)

1. If S spans $M_{2,2}$, then for every 2×2 matrix $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, there exist constants α, β, γ so that
$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \alpha \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \beta \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$
Applying Definition ME, this leads to the linear system
$$\begin{aligned} \alpha + 2\beta &= x \\ 2\alpha + \beta + \gamma &= y \\ 2\alpha - \beta + \gamma &= z \\ \alpha + 2\beta + 2\gamma &= w. \end{aligned}$$
We need to row-reduce the augmented matrix of this system by hand due to the symbols x, y, z, and w in the vector of constants.
$$\begin{bmatrix} 1 & 2 & 0 & x \\ 2 & 1 & 1 & y \\ 2 & -1 & 1 & z \\ 1 & 2 & 2 & w \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & x - y + z \\ 0 & \boxed{1} & 0 & \frac{1}{2}(y - z) \\ 0 & 0 & \boxed{1} & \frac{1}{2}(w - x) \\ 0 & 0 & 0 & \frac{1}{2}(5y - 3x - 3z - w) \end{bmatrix}$$
With the apperance of a leading 1 possible in the last column, by Theorem RCLS there will exist some matrices $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ so that the linear system above has no solution (namely, whenever $5y - 3x - 3z - w \neq 0$), so the set S does not span $M_{2,2}$. (For example, you can verify that there is no solution when $B = \begin{bmatrix} 3 & 3 \\ 3 & 2 \end{bmatrix}$.)

2. To check for linear independence, we need to see if there are nontrivial coefficients α, β, γ that solve
$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \beta \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$
This requires the same work that was done in part (a), with $x = y = z = w = 0$. In that case, the coefficient matrix row-reduces to have a leading 1 in each of the first three columns and a row of zeros on the bottom, so we know that the only solution to the matrix equation is $\alpha = \beta = \gamma = 0$. So the set S is linearly independent.

C26 (Chris Black) Let
$$S = \left\{ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} \right\}$$

1. Determine if S spans $M_{2,2}$.

2. Determine if S is linearly independent.

Solution (Chris Black)

1. The matrices in S will span $M_{2,2}$ if for any $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$, there are coefficients a, b, c, d, e so that
$$a \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + b \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} + c \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} + d \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + e \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$
Thus, we have
$$\begin{bmatrix} a + 2b + d + e & 2a + b + c + 4e \\ 2a - b + c + d & a + 2b + 2c + d + 3e \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

so we have the matrix equation

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 4 \\ 2 & -1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

This system will have a solution for *every* vector on the right side if the row-reduced coefficient matrix has a leading one in every row, since then it is never possible to have a leading 1 appear in the final column of a row-reduced augmented matrix.

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 4 \\ 2 & -1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 0 & 1 \\ 0 & 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & -2 \end{bmatrix}$$

Since there is a leading one in each row of the row-reduced coefficient matrix, there is a solution for every vector $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$, which means that there is a solution to the original equation for every matrix $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$. Thus, the original five matrices span $M_{2,2}$.

2. The matrices in S are linearly independent if the only solution to

$$a \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + b \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} + c \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} + d \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + e \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is $a = b = c = d = e = 0$.

We have

$$\begin{bmatrix} a + 2b + d + e & 2a + b + c + 4e \\ 2a - b + c + d & a + 2b + 2c + d + 3e \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 4 \\ 2 & -1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so we need to find the nullspace of the matrix

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 4 \\ 2 & -1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 & 3 \end{bmatrix}$$

We row-reduced this matrix in part (a), and found that there is a column without a leading 1, which correspons to a free variable in a description of the solution set to the homogeneous system, so the nullspace is nontrivial and there are an infinite number of solutions to

$$a \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + b \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} + c \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} + d \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + e \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, this set of matrices is not linearly independent.

C30 (Robert Beezer) In Example LIM32, find another nontrivial relation of linear dependence on the linearly dependent set of 3×2 matrices, S.

C40 (Robert Beezer) Determine if the set $T = \left\{x^2 - x + 5,\, 4x^3 - x^2 + 5x,\, 3x + 2\right\}$ spans the vector space of polynomials with degree 4 or less, P_4.

Solution (Robert Beezer) The polynomial x^4 is an element of P_4. Can we write this element as a linear combination of the elements of T? To wit, are there scalars a_1, a_2, a_3 such that

$$x^4 = a_1 \left(x^2 - x + 5\right) + a_2 \left(4x^3 - x^2 + 5x\right) + a_3 \left(3x + 2\right)$$

Massaging the right side of this equation, according to the definitions of Example VSP, and then equating coefficients, leads to an inconsistent system of equations (check this!). As such, T is not a spanning set for P_4.

C41 (Robert Beezer) The set W is a subspace of M_{22}, the vector space of all 2×2 matrices. Prove that S is a spanning set for W.

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, 2a - 3b + 4c - d = 0 \right\} \qquad S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix} \right\}$$

Solution (Robert Beezer) We want to show that $W = \langle S \rangle$ (Definition SSVS), which is an equality of sets (Definition SE).

First, show that $\langle S \rangle \subseteq W$. Begin by checking that each of the three matrices in S is a member of the set W. Then, since W is a vector space, the closure properties (Property AC, Property SC) guarantee that every linear combination of elements of S remains in W.

Second, show that $W \subseteq \langle S \rangle$. We want to convince ourselves that an arbitrary element of W is a linear combination of elements of S. Choose

$$\mathbf{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$$

The values of a, b, c, d are not totally arbitrary, since membership in W requires that $2a - 3b + 4c - d = 0$. Now, rewrite as follows,

$$\begin{aligned}
\mathbf{x} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= \begin{bmatrix} a & b \\ c & 2a - 3b + 4c \end{bmatrix} && 2a - 3b + 4c - d = 0 \\
&= \begin{bmatrix} a & 0 \\ 0 & 2a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & -3b \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 4c \end{bmatrix} && \text{Definition MA} \\
&= a \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix} && \text{Definition MSM} \\
&\in \langle S \rangle && \text{Definition SS}
\end{aligned}$$

C42 (Robert Beezer) Determine if the set $S = \{(3,\, 1),\, (7,\, 3)\}$ spans the crazy vector space C (Example CVS).

Solution (Robert Beezer) We will try to show that S spans C. Let $(x,\, y)$ be an arbitrary element of C and search for scalars a_1 and a_2 such that

$$\begin{aligned}
(x,\, y) &= a_1(3,\, 1) + a_2(7,\, 3) \\
&= (4a_1 - 1,\, 2a_1 - 1) + (8a_2 - 1,\, 4a_2 - 1) \\
&= (4a_1 + 8a_2 - 1, 2a_1 + 4a_2 - 1)
\end{aligned}$$

Equality in C leads to the system

$$\begin{aligned}
4a_1 + 8a_2 &= x + 1 \\
2a_1 + 4a_2 &= y + 1
\end{aligned}$$

This system has a singular coefficient matrix whose column space is simply $\left\langle \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\rangle$. So any choice of x and y that causes the column vector $\begin{bmatrix} x+1 \\ y+1 \end{bmatrix}$ to lie outside the column space will lead to an inconsistent system, and hence create an element $(x,\, y)$ that is not in the span of S. So S does not span C.

For example, choose $x = 0$ and $y = 5$, and then we can see that $\begin{bmatrix} 1 \\ 6 \end{bmatrix} \notin \left\langle \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\rangle$ and we know that $(0,\, 5)$ cannot be written as a linear combination of the vectors in S. A shorter solution might begin by asserting that $(0,\, 5)$ is not in $\langle S \rangle$ and then establishing this claim alone.

M10 (Robert Beezer) Halfway through Example SSP4, we need to show that the system of equations

$$\mathcal{LS}\left(\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -8 \\ 0 & 1 & -6 & 24 \\ 1 & -4 & 12 & -32 \\ -2 & 4 & -8 & 16 \end{bmatrix},\ \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}\right)$$

is consistent for every choice of the vector of constants satisfying $16a + 8b + 4c + 2d + e = 0$.

Express the column space of the coefficient matrix of this system as a null space, using Theorem FS. From this use Theorem CSCS to establish that the system is always consistent. Notice that this approach removes from Example SSP4 the need to row-reduce a symbolic matrix.

Solution (Robert Beezer) Theorem FS provides the matrix

$$L = \begin{bmatrix} \boxed{1} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} \end{bmatrix}$$

and so if A denotes the coefficient matrix of the system, then $\mathcal{C}(A) = \mathcal{N}(L)$. The single homogeneous equation in $\mathcal{LS}(L,\ \mathbf{0})$ is equivalent to the condition on the vector of constants (use a, b, c, d, e as variables and then multiply by 16).

T20 (Robert Beezer) Suppose that S is a finite linearly independent set of vectors from the vector space V. Let T be any subset of S. Prove that T is linearly independent.

Solution (Robert Beezer) We will prove the contrapositive (Proof Technique CP): If T is linearly dependent, then S is linearly dependent. This might be an interesting statement in its own right.

Write $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \ldots,\ \mathbf{v}_m\}$ and without loss of generality we can assume that the subset T is the first t vectors of S, $t \leq m$, so $T = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \ldots,\ \mathbf{v}_t\}$. Since T is linearly dependent, by Definition LI there are scalars, not all zero, a_1, a_2, a_3, $\ldots$, a_t, so that

$$\begin{aligned} \mathbf{0} &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_t\mathbf{v}_t \\ &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_t\mathbf{v}_t + \mathbf{0} + \mathbf{0} + \cdots + \mathbf{0} \\ &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_t\mathbf{v}_t + 0\mathbf{v}_{t+1} + 0\mathbf{v}_{t+2} + \cdots + 0\mathbf{v}_m \end{aligned}$$

which is a nontrivial relation of linear dependence (Definition RLD) on the set S, so we can say S is linearly dependent.

T40 (Robert Beezer) Prove the following variant of Theorem EMMVP that has a weaker hypothesis: Suppose that $C = \{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \ldots,\ \mathbf{u}_p\}$ is a linearly independent spanning set for $\mathbb{C}^n$. Suppose also that A and B are $m \times n$ matrices such that $A\mathbf{u}_i = B\mathbf{u}_i$ for every $1 \leq i \leq n$. Then $A = B$.

Can you weaken the hypothesis even further while still preserving the conclusion?

T50 (Robert Beezer) Suppose that V is a vector space and $\mathbf{u}, \mathbf{v} \in V$ are two vectors in V. Use the definition of linear independence to prove that $S = \{\mathbf{u},\ \mathbf{v}\}$ is a linearly dependent set if and only if one of the two vectors is a scalar multiple of the other. Prove this directly in the context of an abstract vector space (V), without simply giving an upgraded version of Theorem DLDS for the special case of just two vectors.

Solution (Robert Beezer) ($\Rightarrow$) If S is linearly dependent, then there are scalars α and β, not both zero, such that $\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{0}$. Suppose that $\alpha \neq 0$, the proof proceeds similarly if $\beta \neq 0$. Now,

$$\begin{aligned} \mathbf{u} &= 1\mathbf{u} && \text{Property O} \\ &= \left(\frac{1}{\alpha}\alpha\right)\mathbf{u} && \text{Property MICN} \\ &= \frac{1}{\alpha}\left(\alpha\mathbf{u}\right) && \text{Property SMA} \\ &= \frac{1}{\alpha}\left(\alpha\mathbf{u} + \mathbf{0}\right) && \text{Property Z} \\ &= \frac{1}{\alpha}\left(\alpha\mathbf{u} + \beta\mathbf{v} - \beta\mathbf{v}\right) && \text{Property AI} \\ &= \frac{1}{\alpha}\left(\mathbf{0} - \beta\mathbf{v}\right) && \text{Definition LI} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\alpha}(-\beta\mathbf{v}) && \text{Property Z} \\ &= \frac{-\beta}{\alpha}\mathbf{v} && \text{Property SMA} \end{aligned}$$

which shows that $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$.

($\Leftarrow$) Suppose now that $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$. More precisely, suppose there is a scalar γ such that $\mathbf{u} = \gamma\mathbf{v}$. Then

$$\begin{aligned} (-1)\mathbf{u} + \gamma\mathbf{v} &= (-1)\mathbf{u} + \mathbf{u} \\ &= (-1)\mathbf{u} + (1)\mathbf{u} && \text{Property O} \\ &= ((-1) + 1)\,\mathbf{u} && \text{Property DSA} \\ &= 0\mathbf{u} && \text{Property AICN} \\ &= \mathbf{0} && \text{Theorem ZSSM} \end{aligned}$$

This is a relation of linear of linear dependence on S (Definition RLD), which is nontrivial since one of the scalars is -1. Therefore S is linearly dependent by Definition LI.

Be careful using this theorem. It is only applicable to sets of two vectors. In particular, linear dependence in a set of three or more vectors can be more complicated than just one vector being a scalar multiple of another.

T51 (Robert Beezer) Carefully formulate the converse of Theorem VRRB and provide a proof.

Solution (Robert Beezer) The converse could read: "Suppose that V is a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_m\}$ is a set of vectors in V. If, for each $\mathbf{w} \in V$, there are *unique* scalars $a_1, a_2, a_3, \ldots, a_m$ such that

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_m\mathbf{v}_m$$

then S is a linearly independent set that spans V."

Since every vector $\mathbf{w} \in V$ is assumed to be a linear combination of the elements of S, it is easy to see that S is a spanning set for V (Definition SSVS).

To establish linear independence, begin with an arbitrary relation of linear dependence on the vectors in S (Definition RLD). One way to form such a relation is the trivial way, where each scalar is zero. But our hypothesis of uniqueness then implies that that the *only* way to form this relation of linear dependence is the trivial way. But this establishes the linear independence of S (Definition LI).

Section B
Bases

C10 (Chris Black) Find a basis for $\langle S\rangle$, where

$$S = \left\{ \begin{bmatrix}1\\3\\2\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\2\\1\end{bmatrix}, \begin{bmatrix}3\\4\\1\\3\end{bmatrix} \right\}.$$

Solution (Chris Black) Theorem BS says that if we take these 5 vectors, put them into a matrix, and row-reduce to discover the pivot columns, then the corresponding vectors in S will be linearly independent and span S, and thus will form a basis of S.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 3 & 2 & 1 & 2 & 4 \\ 2 & 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 & 0 & -2 \\ 0 & \boxed{1} & 2 & 0 & 5 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the independent vectors that span S are the first, second and fourth of the set, so a basis of S is

$$B = \left\{ \begin{bmatrix}1\\3\\2\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\2\\1\end{bmatrix} \right\}$$

C11 (Chris Black) Find a basis for the subspace W of $\mathbb{C}^4$,

$$W = \left\{ \begin{bmatrix} a+b-2c \\ a+b-2c+d \\ -2a+2b+4c-d \\ b+d \end{bmatrix} \middle| \, a, b, c, d \in \mathbb{C} \right\}$$

Solution (Chris Black) We can rewrite an arbitrary vector of W as

$$\begin{aligned}\begin{bmatrix} a+b-2c \\ a+b-2c+d \\ -2a+2b+4c-d \\ b+d \end{bmatrix} &= \begin{bmatrix}a\\a\\-2a\\0\end{bmatrix} + \begin{bmatrix}b\\b\\2b\\b\end{bmatrix} + \begin{bmatrix}-2c\\-2c\\4c\\0\end{bmatrix} + \begin{bmatrix}0\\d\\-d\\d\end{bmatrix} \\ &= a\begin{bmatrix}1\\1\\-2\\0\end{bmatrix} + b\begin{bmatrix}1\\1\\2\\1\end{bmatrix} + c\begin{bmatrix}-2\\-2\\4\\0\end{bmatrix} + d\begin{bmatrix}0\\1\\-1\\1\end{bmatrix}\end{aligned}$$

Thus, we can write W as

$$W = \left\langle \left\{ \begin{bmatrix}1\\1\\-2\\0\end{bmatrix}, \begin{bmatrix}1\\1\\2\\1\end{bmatrix}, \begin{bmatrix}-2\\-2\\4\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\1\end{bmatrix} \right\} \right\rangle$$

These four vectors span W, but we also need to determine if they are linearly independent (turns out they are not). With an application of Theorem BS we can see that the arrive at a basis employing three of these vectors,

$$\begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & 1 & -2 & 1 \\ -2 & 2 & 4 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -2 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, we have the following basis of W,

$$B = \left\{ \begin{bmatrix}1\\1\\-2\\0\end{bmatrix}, \begin{bmatrix}1\\1\\2\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\1\end{bmatrix} \right\}$$

C12 (Chris Black) Find a basis for the vector space T of lower triangular 3×3 matrices; that is, matrices of the form $\begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}$ where an asterisk represents any complex number.

Solution (Chris Black) Let A be an arbitrary element of the specified vector space T. Then there exist a, b, c, d, e and f so that $A = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$. Then

$$A = a\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} + b\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix} + c\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} + d\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix} + e\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix} + f\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

Consider the set

$$B = \left\{ \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} \right\}$$

The six vectors in B span the vector space T, and we can check rather simply that they are also linearly independent. Thus, B is a basis of T.

C13 (Chris Black) Find a basis for the subspace Q of P_2, $Q = \left\{ p(x) = a + bx + cx^2 \middle| p(0) = 0 \right\}$.

Solution (Chris Black) If $p(0) = 0$, then $a + b(0) + c(0^2) = 0$, so $a = 0$. Thus, we can write $Q = \left\{ p(x) = bx + cx^2 \middle| b, c \in \mathbb{C} \right\}$. A linearly independent set that spans Q is $B = \left\{ x, x^2 \right\}$, and this set forms a basis of Q.

C14 (Chris Black) Find a basis for the subspace R of P_2, $R = \left\{ p(x) = a + bx + cx^2 \middle| p'(0) = 0 \right\}$, where p' denotes the derivative.

Solution (Chris Black) The derivative of $p(x) = a + bx + cx^2$ is $p'(x) = b + 2cx$. Thus, if $p \in R$, then $p'(0) = b + 2c(0) = 0$, so we must have $b = 0$. We see that we can rewrite R as $R = \left\{ p(x) = a + cx^2 \middle| a, c \in \mathbb{C} \right\}$. A linearly independent set that spans R is $B = \left\{ 1, x^2 \right\}$, and B is a basis of R.

C40 (Robert Beezer) From Example RSB, form an arbitrary (and nontrivial) linear combination of the four vectors in the original spanning set for W. So the result of this computation is of course an element of W. As such, this vector should be a linear combination of the basis vectors in B. Find the (unique) scalars that provide this linear combination. Repeat with another linear combination of the original four vectors.

Solution (Robert Beezer) An arbitrary linear combination is

$$\mathbf{y} = 3 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 7 \\ -5 \\ 4 \end{bmatrix} + (-2) \begin{bmatrix} -7 \\ -6 \\ -5 \end{bmatrix} = \begin{bmatrix} 25 \\ -10 \\ 15 \end{bmatrix}$$

(You probably used a different collection of scalars.) We want to write $\mathbf{y}$ as a linear combination of

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ \frac{7}{11} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{1}{11} \end{bmatrix} \right\}$$

We could set this up as vector equation with variables as scalars in a linear combination of the vectors in B, but since the first two slots of B have such a nice pattern of zeros and ones, we can determine the necessary scalars easily and then double-check our answer with a computation in the third slot,

$$25 \begin{bmatrix} 1 \\ 0 \\ \frac{7}{11} \end{bmatrix} + (-10) \begin{bmatrix} 0 \\ 1 \\ \frac{1}{11} \end{bmatrix} = \begin{bmatrix} 25 \\ -10 \\ (25)\frac{7}{11} + (-10)\frac{1}{11} \end{bmatrix} = \begin{bmatrix} 25 \\ -10 \\ 15 \end{bmatrix} = \mathbf{y}$$

Notice how the uniqueness of these scalars arises. They are *forced* to be 25 and -10.

C80 (Robert Beezer) Prove that $\{(1, 2), (2, 3)\}$ is a basis for the crazy vector space C (Example CVS).

M20 (Robert Beezer) In Example BM provide the verifications (linear independence and spanning) to show that B is a basis of M_{mn}.

Solution (Robert Beezer) We need to establish the linear independence and spanning properties of the set

$$B = \left\{ B_{k\ell} \middle| 1 \leq k \leq m,\ 1 \leq \ell \leq n \right\}$$

relative to the vector space M_{mn}.

This proof is more transparent if you write out individual matrices in the basis with lots of zeros and dots and a lone one. But we don't have room for that here, so we will use summation notation. Think carefully about each step, especially when the double summations seem to "disappear." Begin with a relation of linear dependence, using double subscripts on the scalars to align with the basis elements.

$$\mathcal{O} = \sum_{k=1}^{m} \sum_{\ell=1}^{n} \alpha_{k\ell} B_{k\ell}$$

Now consider the entry in row i and column j for these equal matrices,

$$\begin{aligned} 0 &= [\mathcal{O}]_{ij} && \text{Definition ZM} \\ &= \left[\sum_{k=1}^{m} \sum_{\ell=1}^{n} \alpha_{k\ell} B_{k\ell} \right]_{ij} && \text{Definition ME} \\ &= \sum_{k=1}^{m} \sum_{\ell=1}^{n} \left[\alpha_{k\ell} B_{k\ell} \right]_{ij} && \text{Definition MA} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{m}\sum_{\ell=1}^{n} \alpha_{k\ell} \left[B_{k\ell}\right]_{ij} && \text{Definition MSM}\\
&= \alpha_{ij} \left[B_{ij}\right]_{ij} && \left[B_{k\ell}\right]_{ij} = 0 \text{ when } (k,\ell) \neq (i,j)\\
&= \alpha_{ij}(1) && \left[B_{ij}\right]_{ij} = 1\\
&= \alpha_{ij}
\end{aligned}$$

Since i and j were arbitrary, we find that each scalar is zero and so B is linearly independent (Definition LI).

To establish the spanning property of B we need only show that an arbitrary matrix A can be written as a linear combination of the elements of B. So suppose that A is an arbitrary $m \times n$ matrix and consider the matrix C defined as a linear combination of the elements of B by

$$C = \sum_{k=1}^{m}\sum_{\ell=1}^{n} \left[A\right]_{k\ell} B_{k\ell}$$

Then,

$$\begin{aligned}
\left[C\right]_{ij} &= \left[\sum_{k=1}^{m}\sum_{\ell=1}^{n} \left[A\right]_{k\ell} B_{k\ell}\right]_{ij} && \text{Definition ME}\\
&= \sum_{k=1}^{m}\sum_{\ell=1}^{n} \left[\left[A\right]_{k\ell} B_{k\ell}\right]_{ij} && \text{Definition MA}\\
&= \sum_{k=1}^{m}\sum_{\ell=1}^{n} \left[A\right]_{k\ell} \left[B_{k\ell}\right]_{ij} && \text{Definition MSM}\\
&= \left[A\right]_{ij} \left[B_{ij}\right]_{ij} && \left[B_{k\ell}\right]_{ij} = 0 \text{ when } (k,\ell) \neq (i,j)\\
&= \left[A\right]_{ij} (1) && \left[B_{ij}\right]_{ij} = 1\\
&= \left[A\right]_{ij}
\end{aligned}$$

So by Definition ME, $A = C$, and therefore $A \in \langle B\rangle$. By Definition B, the set B is a basis of the vector space M_{mn}.

T50 (Robert Beezer) Theorem UMCOB says that unitary matrices are characterized as those matrices that "carry" orthonormal bases to orthonormal bases. This problem asks you to prove a similar result: nonsingular matrices are characterized as those matrices that "carry" bases to bases.

More precisely, suppose that A is a square matrix of size n and $B = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n\}$ is a basis of $\mathbb{C}^n$. Prove that A is nonsingular if and only if $C = \{A\mathbf{x}_1, A\mathbf{x}_2, A\mathbf{x}_3, \dots, A\mathbf{x}_n\}$ is a basis of $\mathbb{C}^n$. (See also Exercise PD.T33, Exercise MR.T20.)

Solution (Robert Beezer) Our first proof relies mostly on definitions of linear independence and spanning, which is a good exercise. The second proof is shorter and turns on a technical result from our work with matrix inverses, Theorem NPNT.

($\Rightarrow$) Assume that A is nonsingular and prove that C is a basis of $\mathbb{C}^n$. First show that C is linearly independent. Work on a relation of linear dependence on C,

$$\begin{aligned}
\mathbf{0} &= a_1A\mathbf{x}_1 + a_2A\mathbf{x}_2 + a_3A\mathbf{x}_3 + \cdots + a_nA\mathbf{x}_n && \text{Definition RLD}\\
&= Aa_1\mathbf{x}_1 + Aa_2\mathbf{x}_2 + Aa_3\mathbf{x}_3 + \cdots + Aa_n\mathbf{x}_n && \text{Theorem MMSMM}\\
&= A\left(a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + a_3\mathbf{x}_3 + \cdots + a_n\mathbf{x}_n\right) && \text{Theorem MMDAA}
\end{aligned}$$

Since A is nonsingular, Definition NM and Theorem SLEMM allows us to conclude that

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n = \mathbf{0}$$

But this is a relation of linear dependence of the linearly independent set B, so the scalars are trivial, $a_1 = a_2 = a_3 = \cdots = a_n = 0$. By Definition LI, the set C is linearly independent.

Now prove that C spans $\mathbb{C}^n$. Given an arbitrary vector $\mathbf{y} \in \mathbb{C}^n$, can it be expressed as a linear combination of the vectors in C? Since A is a nonsingular matrix we can define the vector $\mathbf{w}$ to be the unique solution of the system $\mathcal{LS}(A,\,\mathbf{y})$ (Theorem NMUS). Since $\mathbf{w}\in\mathbb{C}^n$ we can write $\mathbf{w}$ as a linear combination of the vectors in the basis B. So there are scalars, $b_1,\,b_2,\,b_3,\,\ldots,\,b_n$ such that

$$\mathbf{w} = b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + b_3\mathbf{x}_3 + \cdots + b_n\mathbf{x}_n$$

Then,

$$\begin{aligned}
\mathbf{y} &= A\mathbf{w} && \text{Theorem SLEMM}\\
&= A\left(b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + b_3\mathbf{x}_3 + \cdots + b_n\mathbf{x}_n\right) && \text{Definition SSVS}\\
&= Ab_1\mathbf{x}_1 + Ab_2\mathbf{x}_2 + Ab_3\mathbf{x}_3 + \cdots + Ab_n\mathbf{x}_n && \text{Theorem MMDAA}\\
&= b_1A\mathbf{x}_1 + b_2A\mathbf{x}_2 + b_3A\mathbf{x}_3 + \cdots + b_nA\mathbf{x}_n && \text{Theorem MMSMM}
\end{aligned}$$

So we can write an arbitrary vector of $\mathbb{C}^n$ as a linear combination of the elements of C. In other words, C spans $\mathbb{C}^n$ (Definition SSVS). By Definition B, the set C is a basis for $\mathbb{C}^n$.

($\Leftarrow$) Assume that C is a basis and prove that A is nonsingular. Let $\mathbf{x}$ be a solution to the homogeneous system $\mathcal{LS}(A,\,\mathbf{0})$. Since B is a basis of $\mathbb{C}^n$ there are scalars, $a_1,\,a_2,\,a_3,\,\ldots,\,a_n$, such that

$$\mathbf{x} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + a_3\mathbf{x}_3 + \cdots + a_n\mathbf{x}_n$$

Then

$$\begin{aligned}
\mathbf{0} &= A\mathbf{x} && \text{Theorem SLEMM}\\
&= A\left(a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + a_3\mathbf{x}_3 + \cdots + a_n\mathbf{x}_n\right) && \text{Definition SSVS}\\
&= Aa_1\mathbf{x}_1 + Aa_2\mathbf{x}_2 + Aa_3\mathbf{x}_3 + \cdots + Aa_n\mathbf{x}_n && \text{Theorem MMDAA}\\
&= a_1A\mathbf{x}_1 + a_2A\mathbf{x}_2 + a_3A\mathbf{x}_3 + \cdots + a_nA\mathbf{x}_n && \text{Theorem MMSMM}
\end{aligned}$$

This is a relation of linear dependence on the linearly independent set C, so the scalars must all be zero, $a_1 = a_2 = a_3 = \cdots = a_n = 0$. Thus,

$$\mathbf{x} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + a_3\mathbf{x}_3 + \cdots + a_n\mathbf{x}_n = 0\mathbf{x}_1 + 0\mathbf{x}_2 + 0\mathbf{x}_3 + \cdots + 0\mathbf{x}_n = \mathbf{0}.$$

By Definition NM we see that A is nonsingular.

Now for a second proof. Take the vectors for B and use them as the columns of a matrix, $G = [\mathbf{x}_1|\mathbf{x}_2|\mathbf{x}_3|\ldots|\mathbf{x}_n]$. By Theorem CNMB, because we have the hypothesis that B is a basis of $\mathbb{C}^n$, G is a nonsingular matrix. Notice that the columns of AG are exactly the vectors in the set C, by Definition MM.

$$\begin{aligned}
A\text{ nonsingular} &\iff AG\text{ nonsingular} && \text{Theorem NPNT}\\
&\iff C\text{ basis for }\mathbb{C}^n && \text{Theorem CNMB}
\end{aligned}$$

That was easy!

T51 (Robert Beezer) Use the result of Exercise B.T50 to build a very concise proof of Theorem CNMB. (Hint: make a judicious choice for the basis B.)

Solution (Robert Beezer) Choose B to be the set of standard unit vectors, a particularly nice basis of $\mathbb{C}^n$ (Theorem SUVB). For a vector $\mathbf{e}_j$ (Definition SUV) from this basis, what is $A\mathbf{e}_j$?

Section D
Dimension

C20 (Robert Beezer) The archetypes listed below are matrices, or systems of equations with coefficient matrices. For each, compute the nullity and rank of the matrix. This information is listed for each archetype (along with the number of columns in the matrix, so as to illustrate Theorem RPNC), and notice how it could have been computed immediately after the determination of the sets D and F associated with the reduced row-echelon form of the matrix.
Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J, Archetype K, Archetype L

C21 (Chris Black) Find the dimension of the subspace $W = \left\{ \left.\begin{bmatrix} a+b \\ a+c \\ a+d \\ d \end{bmatrix} \right| a, b, c, d \in \mathbb{C} \right\}$ of $\mathbb{C}^4$.

Solution (Chris Black) The subspace W can be written as

$$\begin{aligned} W &= \left\{ \left.\begin{bmatrix} a+b \\ a+c \\ a+d \\ d \end{bmatrix} \right| a, b, c, d \in \mathbb{C} \right\} \\ &= \left\{ \left. a\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right| a, b, c, d \in \mathbb{C} \right\} \\ &= \left\langle \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\} \right\rangle \end{aligned}$$

Since the set of vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a linearly independent set (why?), it forms a basis of W. Thus, W is a subspace of $\mathbb{C}^4$ with dimension 4 (and must therefore equal $\mathbb{C}^4$).

C22 (Chris Black) Find the dimension of the subspace $W = \left\{ a + bx + cx^2 + dx^3 \middle| a + b + c + d = 0 \right\}$ of P_3.

Solution (Chris Black) The subspace $W = \left\{ a + bx + cx^2 + dx^3 \middle| a + b + c + d = 0 \right\}$ can be written as

$$\begin{aligned} W &= \left\{ a + bx + cx^2 + (-a-b-c)x^3 \middle| a, b, c \in \mathbb{C} \right\} \\ &= \left\{ a(1-x^3) + b(x-x^3) + c(x^2-x^3) \middle| a, b, c \in \mathbb{C} \right\} \\ &= \left\langle \left\{ 1-x^3, x-x^3, x^2-x^3 \right\} \right\rangle \end{aligned}$$

Since these vectors are linearly independent (why?), W is a subspace of P_3 with dimension 3.

C23 (Chris Black) Find the dimension of the subspace $W = \left\{ \left.\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| a + b = c, b + c = d, c + d = a \right\}$ of $M_{2,2}$.

Solution (Chris Black) The equations specified are equivalent to the system

$$\begin{aligned} a + b - c &= 0 \\ b + c - d &= 0 \\ a - c - d &= 0 \end{aligned}$$

The coefficient matrix of this system row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 0 & -3 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & -2 \end{bmatrix}$$

Thus, every solution can be decribed with a suitable choice of d, together with $a = 3d$, $b = -d$ and $c = 2d$. Thus the subspace W can be described as

$$W = \left\{ \left.\begin{bmatrix} 3d & -d \\ 2d & d \end{bmatrix} \right| d \in \mathbb{C} \right\} = \left\langle \left\{ \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \right\} \right\rangle$$

So, W is a subspace of $M_{2,2}$ with dimension 1.

C30 (Robert Beezer) For the matrix A below, compute the dimension of the null space of A, $\dim(\mathcal{N}(A))$.

$$A = \begin{bmatrix} 2 & -1 & -3 & 11 & 9 \\ 1 & 2 & 1 & -7 & -3 \\ 3 & 1 & -3 & 6 & 8 \\ 2 & 1 & 2 & -5 & -3 \end{bmatrix}$$

Solution (Robert Beezer) Row reduce A,

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 1 & 1 \\ 0 & \boxed{1} & 0 & -3 & -1 \\ 0 & 0 & \boxed{1} & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So $r = 3$ for this matrix. Then

$$\begin{aligned} \dim(\mathcal{N}(A)) &= n(A) && \text{Definition NOM} \\ &= (n(A) + r(A)) - r(A) \\ &= 5 - r(A) && \text{Theorem RPNC} \\ &= 5 - 3 && \text{Theorem CRN} \\ &= 2 \end{aligned}$$

We could also use Theorem BNS and create a basis for $\mathcal{N}(A)$ with $n - r = 5 - 3 = 2$ vectors (because the solutions are described with 2 free variables) and arrive at the dimension as the size of this basis.

C31 (Robert Beezer) The set W below is a subspace of $\mathbb{C}^4$. Find the dimension of W.

$$W = \left\langle \left\{ \begin{bmatrix} 2 \\ -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ 2 \\ 5 \end{bmatrix} \right\} \right\rangle$$

Solution (Robert Beezer) We will appeal to Theorem BS (or you could consider this an appeal to Theorem BCS). Put the three column vectors of this spanning set into a matrix as columns and row-reduce.

$$\begin{bmatrix} 2 & 3 & -4 \\ -3 & 0 & -3 \\ 4 & 1 & 2 \\ 1 & -2 & 5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are $D = \{1, 2\}$ so we can "keep" the vectors corresponding to the pivot columns and set

$$T = \left\{ \begin{bmatrix} 2 \\ -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}$$

and conclude that $W = \langle T \rangle$ and T is linearly independent. In other words, T is a basis with two vectors, so W has dimension 2.

C35 (Chris Black) Find the rank and nullity of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

Solution (Chris Black) The row reduced form of matrix A is $\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so the rank of A (number of columns with leading 1's) is 3, and the nullity is 0.

C36 (Chris Black) Find the rank and nullity of the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 3 & 2 & 0 & 4 \\ 1 & 2 & 1 & 1 & 1 \end{bmatrix}$.

Solution (Chris Black) The row reduced form of matrix A is $\begin{bmatrix} \boxed{1} & 0 & -1 & 3 & 5 \\ 0 & \boxed{1} & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, so the rank of A (number of columns with leading 1's) is 2, and the nullity is $5 - 2 = 3$.

C37 (Chris Black) Find the rank and nullity of the matrix $A = \begin{bmatrix} 3 & 2 & 1 & 1 & 1 \\ 2 & 3 & 0 & 1 & 1 \\ -1 & 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 & -1 \end{bmatrix}$.

Solution (Chris Black) This matrix A row reduces to the 5×5 identity matrix, so it has full rank. The rank of A is 5, and the nullity is 0.

C40 (Robert Beezer) In Example LDP4 we determined that the set of five polynomials, T, is linearly dependent by a simple invocation of Theorem SSLD. Prove that T is linearly dependent from scratch, beginning with Definition LI.

M20 (Robert Beezer) M_{22} is the vector space of 2×2 matrices. Let S_{22} denote the set of all 2×2 symmetric matrices. That is

$$S_{22} = \left\{ A \in M_{22} |\, A^t = A \right\}$$

1. Show that S_{22} is a subspace of M_{22}.
2. Exhibit a basis for S_{22} and prove that it has the required properties.
3. What is the dimension of S_{22}?

Solution (Robert Beezer) (1) We will use the three criteria of Theorem TSS. The zero vector of M_{22} is the zero matrix, $\mathcal{O}$ (Definition ZM), which is a symmetric matrix. So S_{22} is not empty, since $\mathcal{O} \in S_{22}$.

Suppose that A and B are two matrices in S_{22}. Then we know that $A^t = A$ and $B^t = B$. We want to know if $A + B \in S_{22}$, so test $A + B$ for membership,

$$\begin{aligned} (A+B)^t &= A^t + B^t && \text{Theorem TMA} \\ &= A + B && A,\, B \in S_{22} \end{aligned}$$

So $A + B$ is symmetric and qualifies for membership in S_{22}.

Suppose that $A \in S_{22}$ and $\alpha \in \mathbb{C}$. Is $\alpha A \in S_{22}$? We know that $A^t = A$. Now check that,

$$\begin{aligned} \alpha A^t &= \alpha A^t && \text{Theorem TMSM} \\ &= \alpha A && A \in S_{22} \end{aligned}$$

So αA is also symmetric and qualifies for membership in S_{22}.

With the three criteria of Theorem TSS fulfilled, we see that S_{22} is a subspace of M_{22}.

(2) An arbitrary matrix from S_{22} can be written as $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$. We can express this matrix as

$$\begin{aligned} \begin{bmatrix} a & b \\ b & d \end{bmatrix} &= \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} \\ &= a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

this equation says that the set

$$T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

spans S_{22}. Is it also linearly independent?

Write a relation of linear dependence on S,

$$\begin{aligned} \mathcal{O} &= a_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} \end{aligned}$$

The equality of these two matrices (Definition ME) tells us that $a_1 = a_2 = a_3 = 0$, and the only relation of linear dependence on T is trivial. So T is linearly independent, and hence is a basis of S_{22}.

(3) The basis T found in part (2) has size 3. So by Definition D, $\dim(S_{22}) = 3$.

M21 (Robert Beezer) A 2×2 matrix B is upper triangular if $[B]_{21} = 0$. Let UT_2 be the set of all 2×2 upper triangular matrices. Then UT_2 is a subspace of the vector space of all 2×2 matrices, M_{22} (you may assume this). Determine the dimension of UT_2 providing *all* of the necessary justifications for your answer.

Solution (Robert Beezer) A typical matrix from UT_2 looks like

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

where a, b, $c \in \mathbb{C}$ are arbitrary scalars. Observing this we can then write

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

which says that

$$R = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a spanning set for UT_2 (Definition SSVS). Is R is linearly independent? If so, it is a basis for UT_2. So consider a relation of linear dependence on R,

$$\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \mathcal{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

From this equation, one rapidly arrives at the conclusion that $\alpha_1 = \alpha_2 = \alpha_3 = 0$. So R is a linearly independent set (Definition LI), and hence is a basis (Definition B) for UT_2. Now, we simply count up the size of the set R to see that the dimension of UT_2 is $\dim(UT_2) = 3$.

Section PD
Properties of Dimension

C10 (Robert Beezer) Example SVP4 leaves several details for the reader to check. Verify these five claims.

C40 (Robert Beezer) Determine if the set $T = \left\{x^2 - x + 5,\, 4x^3 - x^2 + 5x,\, 3x + 2\right\}$ spans the vector space of polynomials with degree 4 or less, P_4. (Compare the solution to this exercise with Solution LISS.C40.)

Solution (Robert Beezer) The vector space P_4 has dimension 5 by Theorem DP. Since T contains only 3 vectors, and $3 < 5$, Theorem G tells us that T does not span P_4.

M50 (Robert Beezer) Mimic Definition DS and construct a reasonable definition of $V = U_1 \oplus U_2 \oplus U_3 \oplus \dots \oplus U_m$.

T05 (Robert Beezer) Trivially, if U and V are two subspaces of W, then $\dim(U) = \dim(V)$. Combine this fact, Theorem PSSD, and Theorem EDYES all into one grand combined theorem. You might look to Theorem PIP stylistic inspiration. (Notice this problem does not ask you to prove anything. It just asks you to roll up three theorems into one compact, logically equivalent statement.)

T10 (Robert Beezer) Prove the following theorem, which could be viewed as a reformulation of parts (3) and (4) of Theorem G, or more appropriately as a corollary of Theorem G (Proof Technique LC).

Suppose V is a vector space and S is a subset of V such that the number of vectors in S equals the dimension of V. Then S is linearly independent if and only if S spans V.

T15 (Robert Beezer) Suppose that A is an $m \times n$ matrix and let $\min(m, n)$ denote the minimum of m and n. Prove that $r(A) \leq \min(m, n)$. (If m and n are two numbers, then $\min(m, n)$ stands for the number that is the smaller of the two. For example $\min(4, 6) = 4$.)

T20 (Robert Beezer) Suppose that A is an $m \times n$ matrix and $\mathbf{b} \in \mathbb{C}^m$. Prove that the linear system $\mathcal{LS}(A,\,\mathbf{b})$ is consistent if and only if $r\left(A\right) = r\left([\,A\,|\,\mathbf{b}]\right)$.

Solution (Robert Beezer) ($\Rightarrow$) Suppose first that $\mathcal{LS}(A,\,\mathbf{b})$ is consistent. Then by Theorem CSCS, $\mathbf{b} \in \mathcal{C}(A)$. This means that $\mathcal{C}(A) = \mathcal{C}([\,A\,|\,\mathbf{b}])$ and so it follows that $r\left(A\right) = r\left([\,A\,|\,\mathbf{b}]\right)$.

($\Leftarrow$) Adding a column to a matrix will only increase the size of its column space, so in all cases, $\mathcal{C}(A) \subseteq \mathcal{C}([\,A\,|\,\mathbf{b}])$. However, if we assume that $r\left(A\right) = r\left([\,A\,|\,\mathbf{b}]\right)$, then by Theorem EDYES we conclude that $\mathcal{C}(A) = \mathcal{C}([\,A\,|\,\mathbf{b}])$. Then $\mathbf{b} \in \mathcal{C}([\,A\,|\,\mathbf{b}]) = \mathcal{C}(A)$ so by Theorem CSCS, $\mathcal{LS}(A,\,\mathbf{b})$ is consistent.

T25 (Robert Beezer) Suppose that V is a vector space with finite dimension. Let W be any subspace of V. Prove that W has finite dimension.

T33 (Robert Beezer) Part of Exercise B.T50 is the half of the proof where we assume the matrix A is nonsingular and prove that a set is basis. In Solution B.T50 we proved directly that the set was both linearly independent and a spanning set. Shorten this part of the proof by applying Theorem G. Be careful, there is one subtlety.

Solution (Robert Beezer) By Theorem DCM we know that $\mathbb{C}^n$ has dimension n. So by Theorem G we need only establish that the set C is linearly independent or a spanning set. However, the hypotheses also require that C be of size n. We assumed that $B = \{\mathbf{x}_1,\,\mathbf{x}_2,\,\mathbf{x}_3,\,\ldots,\,\mathbf{x}_n\}$ had size n, but there is no guarantee that $C = \{A\mathbf{x}_1,\,A\mathbf{x}_2,\,A\mathbf{x}_3,\,\ldots,\,A\mathbf{x}_n\}$ will have size n. There could be some "collapsing" or "collisions."

Suppose we establish that C is linearly independent. Then C must have n distinct elements or else we could fashion a nontrivial relation of linear dependence involving duplicate elements.

If we instead to choose to prove that C is a spanning set, then we could establish the uniqueness of the elements of C quite easily. Suppose that $A\mathbf{x}_i = A\mathbf{x}_j$. Then

$$A(\mathbf{x}_i - \mathbf{x}_j) = A\mathbf{x}_i - A\mathbf{x}_j = \mathbf{0}$$

Since A is nonsingular, we conclude that $\mathbf{x}_i - \mathbf{x}_j = \mathbf{0}$, or $\mathbf{x}_i = \mathbf{x}_j$, contrary to our description of B.

T60 (Joe Riegsecker) Suppose that W is a vector space with dimension 5, and U and V are subspaces of W, each of dimension 3. Prove that $U \cap V$ contains a non-zero vector. State a more general result.

Solution (Robert Beezer) Let $\{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3\}$ and $\{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3\}$ be bases for U and V (respectively). Then, the set $\{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3\}$ is linearly dependent, since Theorem G says we cannot have 6 linearly independent vectors in a vector space of dimension 5. So we can assert that there is a non-trivial relation of linear dependence,

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3 = \mathbf{0}$$

where $a_1,\,a_2,\,a_3$ and $b_1,\,b_2,\,b_3$ are not all zero.

We can rearrange this equation as

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 = -b_1\mathbf{v}_1 - b_2\mathbf{v}_2 - b_3\mathbf{v}_3$$

This is an equality of two vectors, so we can give this common vector a name, say $\mathbf{w}$,

$$\mathbf{w} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 = -b_1\mathbf{v}_1 - b_2\mathbf{v}_2 - b_3\mathbf{v}_3$$

This is the desired non-zero vector, as we will now show.

First, since $\mathbf{w} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3$, we can see that $\mathbf{w} \in U$. Similarly, $\mathbf{w} = -b_1\mathbf{v}_1 - b_2\mathbf{v}_2 - b_3\mathbf{v}_3$, so $\mathbf{w} \in V$. This establishes that $\mathbf{w} \in U \cap V$ (Definition SI).

Is $\mathbf{w} \neq \mathbf{0}$? Suppose not, in other words, suppose $\mathbf{w} = \mathbf{0}$. Then

$$\mathbf{0} = \mathbf{w} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3$$

Because $\{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3\}$ is a basis for U, it is a linearly independent set and the relation of linear dependence above means we must conclude that $a_1 = a_2 = a_3 = 0$. By a similar process, we would conclude that $b_1 = b_2 = b_3 = 0$. But this is a contradiction since $a_1,\,a_2,\,a_3,\,b_1,\,b_2,\,b_3$ were chosen so that some were nonzero. So $\mathbf{w} \neq \mathbf{0}$.

How does this generalize? All we really needed was the original relation of linear dependence that resulted because we had "too many" vectors in W. A more general statement would be: Suppose that W is a vector space with dimension n, U is a subspace of dimension p and V is a subspace of dimension q. If $p + q > n$, then $U \cap V$ contains a non-zero vector.

Chapter D
Determinants

Section DM
Determinant of a Matrix

C21 (Chris Black) Doing the computations by hand, find the determinant of the matrix below.

$$\begin{bmatrix} 1 & 3 \\ 6 & 2 \end{bmatrix}$$

Solution (Chris Black) Using the formula in Theorem DMST we have

$$\begin{vmatrix} 1 & 3 \\ 6 & 2 \end{vmatrix} = 1 \cdot 2 - 6 \cdot 3 = 2 - 18 = -16$$

C22 (Chris Black) Doing the computations by hand, find the determinant of the matrix below.

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

Solution (Chris Black) Using the formula in Theorem DMST we have

$$\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 1 \cdot 6 - 2 \cdot 3 = 6 - 6 = 0$$

C23 (Chris Black) Doing the computations by hand, find the determinant of the matrix below.

$$\begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

Solution (Chris Black) We can compute the determinant by expanding about any row or column; the most efficient ones to choose are either the second column or the third row. In any case, the determinant will be -4.

C24 (Robert Beezer) Doing the computations by hand, find the determinant of the matrix below.

$$\begin{bmatrix} -2 & 3 & -2 \\ -4 & -2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$$

Solution (Robert Beezer) We'll expand about the first row since there are no zeros to exploit,

$$\begin{aligned} \begin{vmatrix} -2 & 3 & -2 \\ -4 & -2 & 1 \\ 2 & 4 & 2 \end{vmatrix} &= (-2)\begin{vmatrix} -2 & 1 \\ 4 & 2 \end{vmatrix} + (-1)(3)\begin{vmatrix} -4 & 1 \\ 2 & 2 \end{vmatrix} + (-2)\begin{vmatrix} -4 & -2 \\ 2 & 4 \end{vmatrix} \\ &= (-2)((-2)(2) - 1(4)) + (-3)((-4)(2) - 1(2)) + (-2)((-4)(4) - (-2)(2)) \\ &= (-2)(-8) + (-3)(-10) + (-2)(-12) = 70 \end{aligned}$$

C25 (Robert Beezer) Doing the computations by hand, find the determinant of the matrix below.

$$\begin{bmatrix} 3 & -1 & 4 \\ 2 & 5 & 1 \\ 2 & 0 & 6 \end{bmatrix}$$

Solution (Robert Beezer) We can expand about any row or column, so the zero entry in the middle of the last row is attractive. Let's expand about column 2. By Theorem DER and Theorem DEC you will get the same result by expanding about a different row or column. We will use Theorem DMST twice.

$$\begin{aligned} \begin{vmatrix} 3 & -1 & 4 \\ 2 & 5 & 1 \\ 2 & 0 & 6 \end{vmatrix} &= (-1)(-1)^{1+2}\begin{vmatrix} 2 & 1 \\ 2 & 6 \end{vmatrix} + (5)(-1)^{2+2}\begin{vmatrix} 3 & 4 \\ 2 & 6 \end{vmatrix} + (0)(-1)^{3+2}\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} \\ &= (1)(10) + (5)(10) + 0 = 60 \end{aligned}$$

C26 (Robert Beezer) Doing the computations by hand, find the determinant of the matrix A.

$$A = \begin{bmatrix} 2 & 0 & 3 & 2 \\ 5 & 1 & 2 & 4 \\ 3 & 0 & 1 & 2 \\ 5 & 3 & 2 & 1 \end{bmatrix}$$

Solution (Robert Beezer) With two zeros in column 2, we choose to expand about that column (Theorem DEC),

$$\begin{aligned} \det(A) &= \begin{vmatrix} 2 & 0 & 3 & 2 \\ 5 & 1 & 2 & 4 \\ 3 & 0 & 1 & 2 \\ 5 & 3 & 2 & 1 \end{vmatrix} \\ &= 0(-1)\begin{vmatrix} 5 & 2 & 4 \\ 3 & 1 & 2 \\ 5 & 2 & 1 \end{vmatrix} + 1(1)\begin{vmatrix} 2 & 3 & 2 \\ 3 & 1 & 2 \\ 5 & 2 & 1 \end{vmatrix} + 0(-1)\begin{vmatrix} 2 & 3 & 2 \\ 5 & 2 & 4 \\ 5 & 2 & 1 \end{vmatrix} + 3(1)\begin{vmatrix} 2 & 3 & 2 \\ 5 & 2 & 4 \\ 3 & 1 & 2 \end{vmatrix} \\ &= (1)\left(2(1(1) - 2(2)) - 3(3(1) - 5(2)) + 2(3(2) - 5(1))\right) + \\ &\qquad (3)\left(2(2(2) - 4(1)) - 3(5(2) - 4(3)) + 2(5(1) - 3(2))\right) \\ &= (-6 + 21 + 2) + (3)(0 + 6 - 2) = 29 \end{aligned}$$

C27 (Chris Black) Doing the computations by hand, find the determinant of the matrix A.

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 2 & -1 & 1 \\ 2 & 1 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Solution (Chris Black) Expanding on the first row, we have

$$\begin{aligned} \begin{vmatrix} 1 & 0 & 1 & 1 \\ 2 & 2 & -1 & 1 \\ 2 & 1 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} &= \begin{vmatrix} 2 & -1 & 1 \\ 1 & 3 & 0 \\ 1 & 0 & 1 \end{vmatrix} - 0 + \begin{vmatrix} 2 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 2 & 2 & -1 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{vmatrix} \\ &= 4 + (-1) - (-1) = 4 \end{aligned}$$

C28 (Chris Black) Doing the computations by hand, find the determinant of the matrix A.

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & -1 & -1 & 1 \\ 2 & 5 & 3 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

Solution (Chris Black) Expanding along the first row, we have

$$\begin{vmatrix} 1 & 0 & 1 & 1 \\ 2 & -1 & -1 & 1 \\ 2 & 5 & 3 & 0 \\ 1 & -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -1 & 1 \\ 5 & 3 & 0 \\ -1 & 0 & 1 \end{vmatrix} - 0 + \begin{vmatrix} 2 & -1 & 1 \\ 2 & 5 & 0 \\ 1 & -1 & 1 \end{vmatrix} - \begin{vmatrix} 2 & -1 & -1 \\ 2 & 5 & 3 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= 5 - 0 + 5 - 10 = 0.$$

C29 (Chris Black) Doing the computations by hand, find the determinant of the matrix A.

$$A = \begin{bmatrix} 2 & 3 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Solution (Chris Black) Expanding along the first column, we have

$$\begin{vmatrix} 2 & 3 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{vmatrix} + 0 + 0 + 0 + 0$$

Now, expanding along the first column again, we have

$$\begin{aligned} &= 2 \left(\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} - 0 + \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{vmatrix} - 0 \right) \\ &= 2\left([1 \cdot 1 \cdot 2 + 2 \cdot 0 \cdot 0 + 3 \cdot 2 \cdot 1 - 0 \cdot 1 \cdot 3 - 1 \cdot 0 \cdot 1 - 2 \cdot 2 \cdot 2] + \right. \\ &\quad \left. [1 \cdot 2 \cdot 2 + 1 \cdot 3 \cdot 0 + 2 \cdot 1 \cdot 1 - 0 \cdot 2 \cdot 2 - 1 \cdot 3 \cdot 1 - 2 \cdot 1 \cdot 1]\right) \\ &= 2([2 + 0 + 6 - 0 - 0 - 8] + [4 + 0 + 2 - 0 - 3 - 2]) \\ &= 2 \end{aligned}$$

C30 (Chris Black) Doing the computations by hand, find the determinant of the matrix A.

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 & 1 \\ 2 & 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 3 & 1 & 1 \\ 2 & 1 & 1 & 2 & 1 \end{bmatrix}$$

Solution (Chris Black) In order to exploit the zeros, let's expand along row 3. We then have

$$\begin{vmatrix} 2 & 1 & 1 & 0 & 1 \\ 2 & 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 3 & 1 & 1 \\ 2 & 1 & 1 & 2 & 1 \end{vmatrix} = (-1)^6 \begin{vmatrix} 2 & 1 & 0 & 1 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 2 & 1 \end{vmatrix} + (-1)^7 \cdot 2 \begin{vmatrix} 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 0 & 3 & 1 \\ 2 & 1 & 1 & 1 \end{vmatrix}$$

Notice that the second matrix here is singular since two rows are identical and thus it cannot row-reduce to an identity matrix. We now have

$$= \begin{vmatrix} 2 & 1 & 0 & 1 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 2 & 1 \end{vmatrix} + 0$$

and now we expand on the first row of the first matrix:

$$\begin{aligned} &= 2 \begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} - \begin{vmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{vmatrix} + 0 - \begin{vmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{vmatrix} \\ &= 2(-3) - (-3) - (-3) = 0 \end{aligned}$$

M10 (Chris Black) Find a value of k so that the matrix $A = \begin{bmatrix} 2 & 4 \\ 3 & k \end{bmatrix}$ has $\det(A) = 0$, or explain why it is not possible.

Solution (Chris Black) There is only one value of k that will make this matrix have a zero determinant.

$$\det(A) = \begin{vmatrix} 2 & 4 \\ 3 & k \end{vmatrix} = 2k - 12$$

so $\det(A) = 0$ only when $k = 6$.

M11 (Chris Black) Find a value of k so that the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 2 & 3 & k \end{bmatrix}$ has $\det(A) = 0$, or explain why it is not possible.

Solution (Chris Black)

$$\det(A) = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 2 & 3 & k \end{vmatrix} = 7 - 4k$$

Thus, $\det(A) = 0$ only when $k = \frac{7}{4}$.

M15 (Chris Black) Given the matrix $B = \begin{bmatrix} 2-x & 1 \\ 4 & 2-x \end{bmatrix}$, find all values of x that are solutions of $\det(B) = 0$.

Solution (Chris Black) Using the formula for the determinant of a 2×2 matrix given in Theorem DMST, we have

$$\det(B) = \begin{vmatrix} 2-x & 1 \\ 4 & 2-x \end{vmatrix} = (2-x)(2-x) - 4 = x^2 - 4x = x(x-4)$$

and thus $\det(B) = 0$ only when $x = 0$ or $x = 4$.

M16 (Chris Black) Given the matrix $B = \begin{bmatrix} 4-x & -4 & -4 \\ 2 & -2-x & -4 \\ 3 & -3 & -4-x \end{bmatrix}$, find all values of x that are solutions of $\det(B) = 0$.

Solution (Chris Black)

$$\det(B) = 8x - 2x^2 - x^3 = -x(x^2 + 2x - 8) = -x(x-2)(x+4)$$

And thus, $\det(B) = 0$ when $x = 0$, $x = 2$, or $x = -4$.

Section PDM
Properties of Determinants of Matrices

C30 (Robert Beezer) Each of the archetypes below is a system of equations with a square coefficient matrix, or is a square matrix itself. Compute the determinant of each matrix, noting how Theorem SMZD indicates when the matrix is singular or nonsingular.

Archetype A, Archetype B, Archetype F, Archetype K, Archetype L

M20 (Robert Beezer) Construct a 3×3 nonsingular matrix and call it A. Then, for each entry of the matrix, compute the corresponding cofactor, and create a new 3×3 matrix full of these cofactors by placing the cofactor of an entry in the same location as the entry it was based on. Once complete, call this matrix C. Compute AC^t. Any observations? Repeat with a new matrix, or perhaps with a 4×4 matrix.

Solution (Robert Beezer) The result of these computations should be a matrix with the value of $\det(A)$ in the diagonal entries and zeros elsewhere. The suggestion of using a nonsingular matrix was partially so that it was obvious that the value of the determinant appears on the diagonal.

This result (which is true in general) provides a method for computing the inverse of a nonsingular matrix. Since $AC^t = \det(A) I_n$, we can multiply by the reciprocal of the determinant (which is nonzero!) and the inverse of A (it exists!) to arrive at an expression for the matrix inverse:

$$A^{-1} = \frac{1}{\det(A)} C^t$$

M30 (Robert Beezer) Construct an example to show that the following statement is not true for all square matrices A and B of the same size: $\det(A + B) = \det(A) + \det(B)$.

T10 (Robert Beezer) Theorem NPNT says that if the product of square matrices AB is nonsingular, then the individual matrices A and B are nonsingular also. Construct a new proof of this result making use of theorems about determinants of matrices.

T15 (Robert Beezer) Use Theorem DRCM to prove Theorem DZRC as a corollary. (See Proof Technique LC.)

T20 (Robert Beezer) Suppose that A is a square matrix of size n and $\alpha \in \mathbb{C}$ is a scalar. Prove that $\det(\alpha A) = \alpha^n \det(A)$.

T25 (Robert Beezer) Employ Theorem DT to construct the second half of the proof of Theorem DRCM (the portion about a multiple of a column).

Chapter E
Eigenvalues

Section EE
Eigenvalues and Eigenvectors

C10 (Chris Black) Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution (Chris Black) Answer: $p_A(x) = -2 - 5x + x^2$

C11 (Chris Black) Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$.

Solution (Chris Black) Answer: $p_A(x) = -5 + 4x^2 - x^3$.

C12 (Chris Black) Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix}$.

Solution (Chris Black) Answer: $p_A(x) = 2 + 2x - 2x^2 - 3x^3 + x^4$.

C19 (Robert Beezer) Find the eigenvalues, eigenspaces, algebraic multiplicities and geometric multiplicities for the matrix below. It is possible to do all these computations by hand, and it would be instructive to do so.

$$C = \begin{bmatrix} -1 & 2 \\ -6 & 6 \end{bmatrix}$$

Solution (Robert Beezer) First compute the characteristic polynomial,

$$\begin{aligned} p_C(x) &= \det(C - xI_2) && \text{Definition CP} \\ &= \begin{vmatrix} -1-x & 2 \\ -6 & 6-x \end{vmatrix} \\ &= (-1-x)(6-x) - (2)(-6) && \text{Theorem DMST} \\ &= x^2 - 5x + 6 \\ &= (x-3)(x-2) \end{aligned}$$

So the eigenvalues of C are the solutions to $p_C(x) = 0$, namely, $\lambda = 2$ and $\lambda = 3$. Each eigenvalue has a factor that appears just once in the characteristic polynomial, so $\alpha_A(2) = 1$ and $\alpha_A(3) = 1$.

To obtain the eigenspaces, construct the appropriate singular matrices and find expressions for the null spaces of these matrices.

$$\lambda = 2$$

$$C - (2)I_2 = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & -\frac{2}{3} \\ 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_C(2) = \mathcal{N}(C - (2)I_2) = \left\langle \left\{ \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} \right\} \right\rangle = \left\langle \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\} \right\rangle$$

$$\lambda = 3$$

$$C - (3)I_2 = \begin{bmatrix} -4 & 2 \\ -6 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_C(3) = \mathcal{N}(C - (3)I_2) = \left\langle \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \right\} \right\rangle = \left\langle \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \right\rangle$$

Each eigenspace has a single basis vector, so the dimensions are both 1 and the geometric multiplicities are $\gamma_A(2) = 1$ and $\gamma_A(3) = 1$.

C20 (Robert Beezer) Find the eigenvalues, eigenspaces, algebraic multiplicities and geometric multiplicities for the matrix below. It is possible to do all these computations by hand, and it would be instructive to do so.

$$B = \begin{bmatrix} -12 & 30 \\ -5 & 13 \end{bmatrix}$$

Solution (Robert Beezer) The characteristic polynomial of B is

$$\begin{aligned} p_B(x) &= \det(B - xI_2) && \text{Definition CP} \\ &= \begin{vmatrix} -12 - x & 30 \\ -5 & 13 - x \end{vmatrix} \\ &= (-12 - x)(13 - x) - (30)(-5) && \text{Theorem DMST} \\ &= x^2 - x - 6 \\ &= (x - 3)(x + 2) \end{aligned}$$

From this we find eigenvalues $\lambda = 3, -2$ with algebraic multiplicities $\alpha_B(3) = 1$ and $\alpha_B(-2) = 1$.

For eigenvectors and geometric multiplicities, we study the null spaces of $B - \lambda I_2$ (Theorem EMNS).

$$\lambda = 3 \qquad B - 3I_2 = \begin{bmatrix} -15 & 30 \\ -5 & 10 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & -2 \\ 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_B(3) = \mathcal{N}(B - 3I_2) = \left\langle \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} \right\rangle$$

$$\lambda = -2 \qquad B + 2I_2 = \begin{bmatrix} -10 & 30 \\ -5 & 15 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & -3 \\ 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_B(-2) = \mathcal{N}(B + 2I_2) = \left\langle \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\} \right\rangle$$

Each eigenspace has dimension one, so we have geometric multiplicities $\gamma_B(3) = 1$ and $\gamma_B(-2) = 1$.

C21 (Robert Beezer) The matrix A below has $\lambda = 2$ as an eigenvalue. Find the geometric multiplicity of $\lambda = 2$ using your calculator only for row-reducing matrices.

$$A = \begin{bmatrix} 18 & -15 & 33 & -15 \\ -4 & 8 & -6 & 6 \\ -9 & 9 & -16 & 9 \\ 5 & -6 & 9 & -4 \end{bmatrix}$$

Solution (Robert Beezer) If $\lambda = 2$ is an eigenvalue of A, the matrix $A - 2I_4$ will be singular, and its null space will be the eigenspace of A. So we form this matrix and row-reduce,

$$A - 2I_4 = \begin{bmatrix} 16 & -15 & 33 & -15 \\ -4 & 6 & -6 & 6 \\ -9 & 9 & -18 & 9 \\ 5 & -6 & 9 & -6 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 3 & 0 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

With two free variables, we know a basis of the null space (Theorem BNS) will contain two vectors. Thus the null space of $A - 2I_4$ has dimension two, and so the eigenspace of $\lambda = 2$ has dimension two also (Theorem EMNS), $\gamma_A(2) = 2$.

C22 (Robert Beezer) Without using a calculator, find the eigenvalues of the matrix B.

$$B = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

Solution (Robert Beezer) The characteristic polynomial (Definition CP) is

$$\begin{aligned} p_B(x) &= \det(B - xI_2) \\ &= \begin{vmatrix} 2-x & -1 \\ 1 & 1-x \end{vmatrix} \\ &= (2-x)(1-x) - (1)(-1) && \text{Theorem DMST} \\ &= x^2 - 3x + 3 \\ &= \left(x - \frac{3+\sqrt{3}i}{2}\right)\left(x - \frac{3-\sqrt{3}i}{2}\right) \end{aligned}$$

where the factorization can be obtained by finding the roots of $p_B(x) = 0$ with the quadratic equation. By Theorem EMRCP the eigenvalues of B are the complex numbers $\lambda_1 = \frac{3+\sqrt{3}i}{2}$ and $\lambda_2 = \frac{3-\sqrt{3}i}{2}$.

C23 (Chris Black) Find the eigenvalues, eigenspaces, algebraic and geometric multiplicities for $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Solution (Chris Black) Eigenvalue: $\lambda = 0$

$\mathcal{E}_A(0) = \left\langle \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \right\rangle$, $\alpha_A(0) = 1$, $\gamma_A(0) = 1$

Eigenvalue: $\lambda = 2$

$\mathcal{E}_A(2) = \left\langle \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \right\rangle$, $\alpha_A(2) = 1$, $\gamma_A(2) = 1$

C24 (Chris Black) Find the eigenvalues, eigenspaces, algebraic and geometric multiplicities for $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

Solution (Chris Black) Eigenvalue: $\lambda = 0$

$\mathcal{E}_A(0) = \left\langle \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$, $\alpha_A(0) = 2$, $\gamma_A(0) = 2$

Eigenvalue: $\lambda = 3$

$\mathcal{E}_A(3) = \left\langle \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\} \right\rangle$, $\alpha_A(3) = 1$, $\gamma_A(3) = 1$

C25 (Chris Black) Find the eigenvalues, eigenspaces, algebraic and geometric multiplicities for the 3×3 identity matrix I_3. Do your results make sense?

Solution (Chris Black) The characteristic polynomial for $A = I_3$ is $p_{I_3}(x) = (1-x)^3$, which has eigenvalue $\lambda = 1$ with algebraic multiplicity $\alpha_A(1) = 3$. Looking for eigenvectors, we find that $A - \lambda I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

The nullspace of this matrix is all of $\mathbb{C}^3$, so that the eigenspace is $\mathcal{E}_{I_3}(1) = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$, and the geometric multiplicity is $\gamma_A(1) = 3$.

Does this make sense? Yes! Every vector $\mathbf{x}$ is a solution to $I_3\mathbf{x} = 1\mathbf{x}$, so every nonzero vector is an eigenvector with eigenvalue 1. Since every vector is unchanged when multiplied by I_3, it makes sense that $\lambda = 1$ is the only eigenvalue.

C26 (Chris Black) For matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, the characteristic polynomial of A is $p_A(\lambda) = (4-x)(1-x)^2$. Find the eigenvalues and corresponding eigenspaces of A.

Solution (Chris Black) Since we are given that the characteristic polynomial of A is $p_A(x) = (4-x)(1-x)^2$, we see that the eigenvalues are $\lambda = 4$ with algebraic multiplicity $\alpha_A(4) = 1$ and $\lambda = 1$ with algebraic multiplicity $\alpha_A(1) = 2$. The corresponding eigenspaces are

$$\mathcal{E}_A(4) = \left\langle \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \right\rangle \qquad \mathcal{E}_A(1) = \left\langle \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\} \right\rangle$$

C27 (Chris Black) For matrix $A = \begin{bmatrix} 0 & 4 & -1 & 1 \\ -2 & 6 & -1 & 1 \\ -2 & 8 & -1 & -1 \\ -2 & 8 & -3 & 1 \end{bmatrix}$, the characteristic polynomial of A is

$$p_A(\lambda) = (x+2)(x-2)^2(x-4).$$

Find the eigenvalues and corresponding eigenspaces of A.

Solution (Chris Black) Since we are given that the characteristic polynomial of A is $p_A(x) = (x+2)(x-2)^2(x-4)$, we see that the eigenvalues are $\lambda = -2$, $\lambda = 2$ and $\lambda = 4$. The eigenspaces are

$$\mathcal{E}_A(-2) = \left\langle \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\} \right\rangle$$

$$\mathcal{E}_A(2) = \left\langle \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\} \right\rangle$$

$$\mathcal{E}_A(4) = \left\langle \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \right\rangle$$

M60 (Robert Beezer) Repeat Example CAEHW by choosing $\mathbf{x} = \begin{bmatrix} 0 \\ 8 \\ 2 \\ 1 \\ 2 \end{bmatrix}$ and then arrive at an eigenvalue and eigenvector of the matrix A. The hard way.

Solution (Robert Beezer) Form the matrix C whose columns are $\mathbf{x}$, $A\mathbf{x}$, $A^2\mathbf{x}$, $A^3\mathbf{x}$, $A^4\mathbf{x}$, $A^5\mathbf{x}$ and row-reduce the matrix,

$$\begin{bmatrix} 0 & 6 & 32 & 102 & 320 & 966 \\ 8 & 10 & 24 & 58 & 168 & 490 \\ 2 & 12 & 50 & 156 & 482 & 1452 \\ 1 & -5 & -47 & -149 & -479 & -1445 \\ 2 & 12 & 50 & 156 & 482 & 1452 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & -3 & -9 & -30 \\ 0 & \boxed{1} & 0 & 1 & 0 & 1 \\ 0 & 0 & \boxed{1} & 3 & 10 & 30 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The simplest possible relation of linear dependence on the columns of C comes from using scalars $\alpha_4 = 1$ and $\alpha_5 = \alpha_6 = 0$ for the free variables in a solution to $\mathcal{LS}(C, \mathbf{0})$. The remainder of this solution is $\alpha_1 = 3$, $\alpha_2 = -1$, $\alpha_3 = -3$. This solution gives rise to the polynomial

$$p(x) = 3 - x - 3x^2 + x^3 = (x-3)(x-1)(x+1)$$

which then has the property that $p(A)\mathbf{x} = \mathbf{0}$.

No matter how you choose to order the factors of $p(x)$, the value of k (in the language of Theorem EMHE and Example CAEHW) is $k = 2$. For each of the three possibilities, we list the resulting eigenvector and the associated eigenvalue:

$$(C - 3I_5)(C - I_5)\mathbf{z} = \begin{bmatrix} 8 \\ 8 \\ 8 \\ -24 \\ 8 \end{bmatrix} \qquad \lambda = -1$$

$$(C - 3I_5)(C + I_5)\mathbf{z} = \begin{bmatrix} 20 \\ -20 \\ 20 \\ -40 \\ 20 \end{bmatrix} \qquad \lambda = 1$$

$$(C + I_5)(C - I_5)\mathbf{z} = \begin{bmatrix} 32 \\ 16 \\ 48 \\ -48 \\ 48 \end{bmatrix} \qquad \lambda = 3$$

Note that each of these eigenvectors can be simplified by an appropriate scalar multiple, but we have shown here the actual vector obtained by the product specified in the theorem.

T10 (Robert Beezer) A matrix A is idempotent if $A^2 = A$. Show that the only possible eigenvalues of an idempotent matrix are $\lambda = 0$ and $\lambda = 1$. Then give an example of a matrix that is idempotent and has both of these two values as eigenvalues.

Solution (Robert Beezer) Suppose that λ is an eigenvalue of A. Then there is an eigenvector $\mathbf{x}$, such that $A\mathbf{x} = \lambda\mathbf{x}$. We have,

$$\begin{aligned}
\lambda\mathbf{x} &= A\mathbf{x} && \mathbf{x} \text{ eigenvector of } A \\
&= A^2\mathbf{x} && A \text{ is idempotent} \\
&= A(A\mathbf{x}) && \\
&= A(\lambda\mathbf{x}) && \mathbf{x} \text{ eigenvector of } A \\
&= \lambda(A\mathbf{x}) && \text{Theorem MMSMM} \\
&= \lambda(\lambda\mathbf{x}) && \mathbf{x} \text{ eigenvector of } A \\
&= \lambda^2\mathbf{x} &&
\end{aligned}$$

From this we get

$$\begin{aligned}
\mathbf{0} &= \lambda^2\mathbf{x} - \lambda\mathbf{x} && \\
&= (\lambda^2 - \lambda)\mathbf{x} && \text{Property DSAC}
\end{aligned}$$

Since $\mathbf{x}$ is an eigenvector, it is nonzero, and Theorem SMEZV leaves us with the conclusion that $\lambda^2 - \lambda = 0$, and the solutions to this quadratic polynomial equation in λ are $\lambda = 0$ and $\lambda = 1$.

The matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is idempotent (check this!) and since it is a diagonal matrix, its eigenvalues are the diagonal entries, $\lambda = 0$ and $\lambda = 1$, so each of these possible values for an eigenvalue of an idempotent matrix actually occurs as an eigenvalue of some idempotent matrix. So we cannot state any stronger conclusion about the eigenvalues of an idempotent matrix, and we can say that this theorem is the "best possible."

T15 (Robert Beezer) The characteristic polynomial of the square matrix A is usually defined as $r_A(x) = \det(xI_n - A)$. Find a specific relationship between our characteristic polynomial, $p_A(x)$, and $r_A(x)$, give a proof of your relationship, and use this to explain why Theorem EMRCP can remain essentially unchanged with either definition. Explain the advantages of each definition over the other. (Computing with both definitions, for a 2×2 and a 3×3 matrix, might be a good way to start.)

Solution (Robert Beezer) Note in the following that the scalar multiple of a matrix is equivalent to multiplying each of the rows by that scalar, so we actually apply Theorem DRCM multiple times below (and are passing up an opportunity to do a proof by induction in the process, which maybe you'd like to do yourself?).

$$\begin{aligned}
p_A(x) &= \det(A - xI_n) && \text{Definition CP} \\
&= \det((-1)(xI_n - A)) && \text{Definition MSM} \\
&= (-1)^n \det(xI_n - A) && \text{Theorem DRCM}
\end{aligned}$$

$$= (-1)^n r_A(x)$$

Since the polynomials are scalar multiples of each other, their roots will be identical, so either polynomial could be used in Theorem EMRCP.

Computing by hand, our definition of the characteristic polynomial is easier to use, as you only need to subtract x down the diagonal of the matrix before computing the determinant. However, the price to be paid is that for odd values of n, the coefficient of x^n is -1, while $r_A(x)$ always has the coefficient 1 for x^n (we say $r_A(x)$ is "monic.")

T20 (Robert Beezer) Suppose that λ and ρ are two different eigenvalues of the square matrix A. Prove that the intersection of the eigenspaces for these two eigenvalues is trivial. That is, $\mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho) = \{\mathbf{0}\}$.

Solution (Robert Beezer) This problem asks you to prove that two sets are equal, so use Definition SE.

First show that $\{\mathbf{0}\} \subseteq \mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho)$. Choose $\mathbf{x} \in \{\mathbf{0}\}$. Then $\mathbf{x} = \mathbf{0}$. Eigenspaces are subspaces (Theorem EMS), so both $\mathcal{E}_A(\lambda)$ and $\mathcal{E}_A(\rho)$ contain the zero vector, and therefore $\mathbf{x} \in \mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho)$ (Definition SI).

To show that $\mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho) \subseteq \{\mathbf{0}\}$, suppose that $\mathbf{x} \in \mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho)$. Then $\mathbf{x}$ is an eigenvector of A for both λ and ρ (Definition SI) and so

$$\begin{aligned}
\mathbf{x} &= 1\mathbf{x} && \text{Property O}\\
&= \frac{1}{\lambda-\rho}(\lambda-\rho)\mathbf{x} && \lambda \neq \rho,\ \lambda-\rho \neq 0\\
&= \frac{1}{\lambda-\rho}(\lambda\mathbf{x}-\rho\mathbf{x}) && \text{Property DSAC}\\
&= \frac{1}{\lambda-\rho}(A\mathbf{x}-A\mathbf{x}) && \mathbf{x}\text{ eigenvector of } A \text{ for } \lambda,\ \rho\\
&= \frac{1}{\lambda-\rho}(\mathbf{0}) &&\\
&= \mathbf{0} && \text{Theorem ZVSM}
\end{aligned}$$

So $\mathbf{x} = \mathbf{0}$, and trivially, $\mathbf{x} \in \{\mathbf{0}\}$.

Section PEE
Properties of Eigenvalues and Eigenvectors

T10 (Robert Beezer) Suppose that A is a square matrix. Prove that the constant term of the characteristic polynomial of A is equal to the determinant of A.

Solution (Robert Beezer) Suppose that the characteristic polynomial of A is

$$p_A(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

Then

$$\begin{aligned}
a_0 &= a_0 + a_1(0) + a_2(0)^2 + \cdots + a_n(0)^n\\
&= p_A(0)\\
&= \det(A - 0I_n) && \text{Definition CP}\\
&= \det(A)
\end{aligned}$$

T20 (Robert Beezer) Suppose that A is a square matrix. Prove that a single vector may not be an eigenvector of A for two different eigenvalues.

Solution (Robert Beezer) Suppose that the vector $\mathbf{x} \neq \mathbf{0}$ is an eigenvector of A for the two eigenvalues λ and ρ, where $\lambda \neq \rho$. Then $\lambda - \rho \neq 0$, and we also have

$$\begin{aligned}
\mathbf{0} &= A\mathbf{x} - A\mathbf{x} && \text{Property AIC}\\
&= \lambda\mathbf{x} - \rho\mathbf{x} && \text{Definition EEM}
\end{aligned}$$

$$= (\lambda - \rho)\mathbf{x} \qquad \text{Property DSAC}$$

By Theorem SMEZV, either $\lambda - \rho = 0$ or $\mathbf{x} = \mathbf{0}$, which are both contradictions.

T22 (Robert Beezer) Suppose that U is a unitary matrix with eigenvalue λ. Prove that λ has modulus 1, i.e. $|\lambda| = 1$. This says that all of the eigenvalues of a unitary matrix lie on the unit circle of the complex plane.

T30 (Robert Beezer) Theorem DCP tells us that the characteristic polynomial of a square matrix of size n has degree n. By suitably augmenting the proof of Theorem DCP prove that the coefficient of x^n in the characteristic polynomial is $(-1)^n$.

T50 (Robert Beezer) Theorem EIM says that if λ is an eigenvalue of the nonsingular matrix A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}. Write an alternate proof of this theorem using the characteristic polynomial and without making reference to an eigenvector of A for λ.

Solution (Robert Beezer) Since λ is an eigenvalue of a nonsingular matrix, $\lambda \neq 0$ (Theorem SMZE). A is invertible (Theorem NI), and so $-\lambda A$ is invertible (Theorem MISM). Thus $-\lambda A$ is nonsingular (Theorem NI) and $\det(-\lambda A) \neq 0$ (Theorem SMZD).

$$\begin{aligned}
p_{A^{-1}}\left(\frac{1}{\lambda}\right) &= \det\left(A^{-1} - \frac{1}{\lambda}I_n\right) && \text{Definition CP}\\
&= 1\det\left(A^{-1} - \frac{1}{\lambda}I_n\right) && \text{Property OCN}\\
&= \frac{1}{\det(-\lambda A)}\det(-\lambda A)\det\left(A^{-1} - \frac{1}{\lambda}I_n\right) && \text{Property MICN}\\
&= \frac{1}{\det(-\lambda A)}\det\left((-\lambda A)\left(A^{-1} - \frac{1}{\lambda}I_n\right)\right) && \text{Theorem DRMM}\\
&= \frac{1}{\det(-\lambda A)}\det\left(-\lambda A A^{-1} - (-\lambda A)\frac{1}{\lambda}I_n\right) && \text{Theorem MMDAA}\\
&= \frac{1}{\det(-\lambda A)}\det\left(-\lambda I_n - (-\lambda A)\frac{1}{\lambda}I_n\right) && \text{Definition MI}\\
&= \frac{1}{\det(-\lambda A)}\det\left(-\lambda I_n + \lambda\frac{1}{\lambda}AI_n\right) && \text{Theorem MMSMM}\\
&= \frac{1}{\det(-\lambda A)}\det\left(-\lambda I_n + 1AI_n\right) && \text{Property MICN}\\
&= \frac{1}{\det(-\lambda A)}\det\left(-\lambda I_n + AI_n\right) && \text{Property OCN}\\
&= \frac{1}{\det(-\lambda A)}\det\left(-\lambda I_n + A\right) && \text{Theorem MMIM}\\
&= \frac{1}{\det(-\lambda A)}\det\left(A - \lambda I_n\right) && \text{Property ACM}\\
&= \frac{1}{\det(-\lambda A)}p_A(\lambda) && \text{Definition CP}\\
&= \frac{1}{\det(-\lambda A)}0 && \text{Theorem EMRCP}\\
&= 0 && \text{Property ZCN}
\end{aligned}$$

So $\frac{1}{\lambda}$ is a root of the characteristic polynomial of A^{-1} and so is an eigenvalue of A^{-1}. This proof is due to Sara Bucht.

Section SD
Similarity and Diagonalization

C20 (Robert Beezer) Consider the matrix A below. First, show that A is diagonalizable by computing the geometric multiplicities of the eigenvalues and quoting the relevant theorem. Second, find a diagonal matrix D and a nonsingular matrix S so that $S^{-1}AS = D$. (See Exercise EE.C20 for some of the necessary computations.)

$$A = \begin{bmatrix} 18 & -15 & 33 & -15 \\ -4 & 8 & -6 & 6 \\ -9 & 9 & -16 & 9 \\ 5 & -6 & 9 & -4 \end{bmatrix}$$

Solution (Robert Beezer) Using a calculator, we find that A has three distinct eigenvalues, $\lambda = 3,\ 2,\ -1$, with $\lambda = 2$ having algebraic multiplicity two, $\alpha_A(2) = 2$. The eigenvalues $\lambda = 3,\ -1$ have algebraic multiplicity one, and so by Theorem ME we can conclude that their geometric multiplicities are one as well. Together with the computation of the geometric multiplicity of $\lambda = 2$ from Exercise EE.C20, we know

$$\gamma_A(3) = \alpha_A(3) = 1 \qquad \gamma_A(2) = \alpha_A(2) = 2 \qquad \gamma_A(-1) = \alpha_A(-1) = 1$$

This satisfies the hypotheses of Theorem DMFE, and so we can conclude that A is diagonalizable.

A calculator will give us four eigenvectors of A, the two for $\lambda = 2$ being linearly independent presumably. Or, by hand, we could find basis vectors for the three eigenspaces. For $\lambda = 3,\ -1$ the eigenspaces have dimension one, and so any eigenvector for these eigenvalues will be multiples of the ones we use below. For $\lambda = 2$ there are many different bases for the eigenspace, so your answer could vary. Our eigenvectors are the basis vectors we would have obtained if we had actually constructed a basis in Exercise EE.C20 rather than just computing the dimension.

By the construction in the proof of Theorem DC, the required matrix S has columns that are four linearly independent eigenvectors of A and the diagonal matrix has the eigenvalues on the diagonal (in the same order as the eigenvectors in S). Here are the pieces, "doing" the diagonalization,

$$\begin{bmatrix} -1 & 0 & -3 & 6 \\ -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & -3 \\ 1 & 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 18 & -15 & 33 & -15 \\ -4 & 8 & -6 & 6 \\ -9 & 9 & -16 & 9 \\ 5 & -6 & 9 & -4 \end{bmatrix} \begin{bmatrix} -1 & 0 & -3 & 6 \\ -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & -3 \\ 1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

C21 (Robert Beezer) Determine if the matrix A below is diagonalizable. If the matrix is diagonalizable, then find a diagonal matrix D that is similar to A, and provide the invertible matrix S that performs the similarity transformation. You should use your calculator to find the eigenvalues of the matrix, but try only using the row-reducing function of your calculator to assist with finding eigenvectors.

$$A = \begin{bmatrix} 1 & 9 & 9 & 24 \\ -3 & -27 & -29 & -68 \\ 1 & 11 & 13 & 26 \\ 1 & 7 & 7 & 18 \end{bmatrix}$$

Solution (Robert Beezer) A calculator will provide the eigenvalues $\lambda = 2,\ 2,\ 1,\ 0$, so we can reconstruct the characteristic polynomial as

$$p_A(x) = (x-2)^2(x-1)x$$

so the algebraic multiplicities of the eigenvalues are

$$\alpha_A(2) = 2 \qquad \alpha_A(1) = 1 \qquad \alpha_A(0) = 1$$

Now compute eigenspaces by hand, obtaining null spaces for each of the three eigenvalues by constructing the correct singular matrix (Theorem EMNS),

$$A - 2I_4 = \begin{bmatrix} -1 & 9 & 9 & 24 \\ -3 & -29 & -29 & -68 \\ 1 & 11 & 11 & 26 \\ 1 & 7 & 7 & 16 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 1 & \frac{5}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_A(2) = \mathcal{N}(A - 2I_4) = \left\langle\left\{ \begin{bmatrix} \frac{3}{2} \\ -\frac{5}{2} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}\right\rangle = \left\langle\left\{ \begin{bmatrix} 3 \\ -5 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}\right\rangle$$

$$A - 1I_4 = \begin{bmatrix} 0 & 9 & 9 & 24 \\ -3 & -28 & -29 & -68 \\ 1 & 11 & 12 & 26 \\ 1 & 7 & 7 & 17 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -\frac{5}{3} \\ 0 & 1 & 0 & \frac{13}{3} \\ 0 & 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_A(1) = \mathcal{N}(A - I_4) = \left\langle\left\{\begin{bmatrix} \frac{5}{3} \\ -\frac{13}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix}\right\}\right\rangle = \left\langle\left\{\begin{bmatrix} 5 \\ -13 \\ 5 \\ 3 \end{bmatrix}\right\}\right\rangle$$

$$A - 0I_4 = \begin{bmatrix} 1 & 9 & 9 & 24 \\ -3 & -27 & -29 & -68 \\ 1 & 11 & 13 & 26 \\ 1 & 7 & 7 & 18 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_A(0) = \mathcal{N}(A - I_4) = \left\langle\left\{\begin{bmatrix} 3 \\ -5 \\ 2 \\ 1 \end{bmatrix}\right\}\right\rangle$$

From this we can compute the dimensions of the eigenspaces to obtain the geometric multiplicities,

$$\gamma_A(2) = 2 \qquad \gamma_A(1) = 1 \qquad \gamma_A(0) = 1$$

For each eigenvalue, the algebraic and geometric multiplicities are equal and so by Theorem DMFE we now know that A is diagonalizable. The construction in Theorem DC suggests we form a matrix whose columns are eigenvectors of A

$$S = \begin{bmatrix} 3 & 0 & 5 & 3 \\ -5 & -1 & -13 & -5 \\ 0 & 1 & 5 & 2 \\ 2 & 0 & 3 & 1 \end{bmatrix}$$

Since $\det(S) = -1 \neq 0$, we know that S is nonsingular (Theorem SMZD), so the columns of S are a set of 4 linearly independent eigenvectors of A. By the proof of Theorem SMZD we know

$$S^{-1}AS = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a diagonal matrix with the eigenvalues of A along the diagonal, in the same order as the associated eigenvectors appear as columns of S.

C22 (Robert Beezer) Consider the matrix A below. Find the eigenvalues of A using a calculator and use these to construct the characteristic polynomial of A, $p_A(x)$. State the algebraic multiplicity of each eigenvalue. Find all of the eigenspaces for A by computing expressions for null spaces, only using your calculator to row-reduce matrices. State the geometric multiplicity of each eigenvalue. Is A diagonalizable? If not, explain why. If so, find a diagonal matrix D that is similar to A.

$$A = \begin{bmatrix} 19 & 25 & 30 & 5 \\ -23 & -30 & -35 & -5 \\ 7 & 9 & 10 & 1 \\ -3 & -4 & -5 & -1 \end{bmatrix}$$

Solution (Robert Beezer) A calculator will report $\lambda = 0$ as an eigenvalue of algebraic multiplicity of 2, and $\lambda = -1$ as an eigenvalue of algebraic multiplicity 2 as well. Since eigenvalues are roots of the characteristic polynomial (Theorem EMRCP) we have the factored version

$$p_A(x) = (x - 0)^2(x - (-1))^2 = x^2(x^2 + 2x + 1) = x^4 + 2x^3 + x^2$$

The eigenspaces are then

$\lambda = 0$

$$A - (0)I_4 = \begin{bmatrix} 19 & 25 & 30 & 5 \\ -23 & -30 & -35 & -5 \\ 7 & 9 & 10 & 1 \\ -3 & -4 & -5 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -5 & -5 \\ 0 & \boxed{1} & 5 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_A(0) = \mathcal{N}(C - (0)I_4) = \left\langle\left\{\begin{bmatrix}5\\-5\\1\\0\end{bmatrix}, \begin{bmatrix}5\\-4\\0\\1\end{bmatrix}\right\}\right\rangle$$

$$\lambda = -1$$

$$A - (-1)I_4 = \begin{bmatrix}20 & 25 & 30 & 5\\-23 & -29 & -35 & -5\\7 & 9 & 11 & 1\\-3 & -4 & -5 & 0\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & -1 & 4\\0 & \boxed{1} & 2 & -3\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

$$\mathcal{E}_A(-1) = \mathcal{N}(C - (-1)I_4) = \left\langle\left\{\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}-4\\3\\0\\1\end{bmatrix}\right\}\right\rangle$$

Each eigenspace above is described by a spanning set obtained through an application of Theorem BNS and so is a basis for the eigenspace. In each case the dimension, and therefore the geometric multiplicity, is 2.

For each of the two eigenvalues, the algebraic and geometric multiplicities are equal. Theorem DMFE says that in this situation the matrix is diagonalizable. We know from Theorem DC that when we diagonalize A the diagonal matrix will have the eigenvalues of A on the diagonal (in some order). So we can claim that

$$D = \begin{bmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & -1 & 0\\0 & 0 & 0 & -1\end{bmatrix}$$

T15 (Robert Beezer) Suppose that A and B are similar matrices. Prove that A^3 and B^3 are similar matrices. Generalize.

Solution (Robert Beezer) By Definition SIM we know that there is a nonsingular matrix S so that $A = S^{-1}BS$. Then

$$\begin{aligned}
A^3 &= (S^{-1}BS)^3\\
&= (S^{-1}BS)(S^{-1}BS)(S^{-1}BS)\\
&= S^{-1}B(SS^{-1})B(SS^{-1})BS && \text{Theorem MMA}\\
&= S^{-1}B(I_3)B(I_3)BS && \text{Definition MI}\\
&= S^{-1}BBBS && \text{Theorem MMIM}\\
&= S^{-1}B^3S
\end{aligned}$$

This equation says that A^3 is similar to B^3 (via the matrix S).

More generally, if A is similar to B, and m is a non-negative integer, then A^m is similar to B^m. This can be proved using induction (Proof Technique I).

T16 (Robert Beezer) Suppose that A and B are similar matrices, with A nonsingular. Prove that B is nonsingular, and that A^{-1} is similar to B^{-1}.

Solution (Steve Canfield) A being similar to B means that there exists an S such that $A = S^{-1}BS$. So, $B = SAS^{-1}$ and because S, A, and S^{-1} are nonsingular, by Theorem NPNT, B is nonsingular.

$$\begin{aligned}
A^{-1} &= \left(S^{-1}BS\right)^{-1} && \text{Definition SIM}\\
&= S^{-1}B^{-1}\left(S^{-1}\right)^{-1} && \text{Theorem SS} \qquad = S^{-1}B^{-1}S \qquad \text{Theorem MIMI}
\end{aligned}$$

Then by Definition SIM, A^{-1} is similar to B^{-1}.

T17 (Robert Beezer) Suppose that B is a nonsingular matrix. Prove that AB is similar to BA.

Solution (Robert Beezer) The nonsingular (invertible) matrix B will provide the desired similarity transformation,

$$B^{-1}(BA)B = \left(B^{-1}B\right)(AB) \qquad \text{Theorem MMA}$$

$$= I_n AB \qquad \text{Definition MI}$$
$$= AB \qquad \text{Theorem MMIM}$$

Chapter LT
Linear Transformations

Section LT
Linear Transformations

C15 (Robert Beezer) The archetypes below are all linear transformations whose domains and codomains are vector spaces of column vectors (Definition VSCV). For each one, compute the matrix representation described in the proof of Theorem MLTCV.

Archetype M, Archetype N, Archetype O, Archetype P, Archetype Q, Archetype R

C16 (Chris Black) Find the matrix representation of $T\colon \mathbb{C}^3 \to \mathbb{C}^4$, $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}3x+2y+z\\x+y+z\\x-3y\\2x+3y+z\end{bmatrix}$.

Solution (Chris Black) Answer: $A_T = \begin{bmatrix}3&2&1\\1&1&1\\1&-3&0\\2&3&1\end{bmatrix}$.

C20 (Robert Beezer) Let $\mathbf{w} = \begin{bmatrix}-3\\1\\4\end{bmatrix}$. Referring to Example MOLT, compute $S(\mathbf{w})$ two different ways. First use the definition of S, then compute the matrix-vector product $C\mathbf{w}$ (Definition MVP).

Solution (Robert Beezer) In both cases the result will be $S(\mathbf{w}) = \begin{bmatrix}9\\2\\-9\\4\end{bmatrix}$.

C25 (Robert Beezer) Define the linear transformation

$$T\colon \mathbb{C}^3 \to \mathbb{C}^2, \quad T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}2x_1 - x_2 + 5x_3\\-4x_1 + 2x_2 - 10x_3\end{bmatrix}$$

Verify that T is a linear transformation.

Solution (Robert Beezer) We can rewrite T as follows:

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}2x_1 - x_2 + 5x_3\\-4x_1 + 2x_2 - 10x_3\end{bmatrix} = x_1\begin{bmatrix}2\\-4\end{bmatrix} + x_2\begin{bmatrix}-1\\2\end{bmatrix} + x_3\begin{bmatrix}5\\-10\end{bmatrix} = \begin{bmatrix}2&-1&5\\-4&2&-10\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

and Theorem MBLT tell us that any function of this form is a linear transformation.

C26 (Robert Beezer) Verify that the function below is a linear transformation.

$$T\colon P_2 \to \mathbb{C}^2, \quad T\left(a + bx + cx^2\right) = \begin{bmatrix}2a-b\\b+c\end{bmatrix}$$

Solution (Robert Beezer) Check the two conditions of Definition LT.

$$\begin{aligned}T(\mathbf{u}+\mathbf{v}) &= T\left(\left(a+bx+cx^2\right)+\left(d+ex+fx^2\right)\right)\\ &= T\left((a+d)+(b+e)\,x+(c+f)\,x^2\right)\\ &= \begin{bmatrix}2(a+d)-(b+e)\\(b+e)+(c+f)\end{bmatrix}\\ &= \begin{bmatrix}(2a-b)+(2d-e)\\(b+c)+(e+f)\end{bmatrix}\\ &= \begin{bmatrix}2a-b\\b+c\end{bmatrix}+\begin{bmatrix}2d-e\\e+f\end{bmatrix}\\ &= T(\mathbf{u})+T(\mathbf{v})\end{aligned}$$

and

$$\begin{aligned}T(\alpha\mathbf{u}) &= T\left(\alpha\left(a+bx+cx^2\right)\right)\\ &= T\left((\alpha a)+(\alpha b)\,x+(\alpha c)\,x^2\right)\\ &= \begin{bmatrix}2(\alpha a)-(\alpha b)\\(\alpha b)+(\alpha c)\end{bmatrix}\\ &= \begin{bmatrix}\alpha(2a-b)\\\alpha(b+c)\end{bmatrix}\\ &= \alpha\begin{bmatrix}2a-b\\b+c\end{bmatrix}\\ &= \alpha T(\mathbf{u})\end{aligned}$$

So T is indeed a linear transformation.

C30 (Robert Beezer) Define the linear transformation

$$T\colon \mathbb{C}^3\to\mathbb{C}^2,\quad T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\begin{bmatrix}2x_1-x_2+5x_3\\-4x_1+2x_2-10x_3\end{bmatrix}$$

Compute the preimages, $T^{-1}\left(\begin{bmatrix}2\\3\end{bmatrix}\right)$ and $T^{-1}\left(\begin{bmatrix}4\\-8\end{bmatrix}\right)$.

Solution (Robert Beezer) For the first pre-image, we want $\mathbf{x}\in\mathbb{C}^3$ such that $T(\mathbf{x})=\begin{bmatrix}2\\3\end{bmatrix}$. This becomes,

$$\begin{bmatrix}2x_1-x_2+5x_3\\-4x_1+2x_2-10x_3\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}$$

Vector equality gives a system of two linear equations in three variables, represented by the augmented matrix

$$\begin{bmatrix}2&-1&5&2\\-4&2&-10&3\end{bmatrix}\xrightarrow{\text{RREF}}\begin{bmatrix}\boxed{1}&-\frac{1}{2}&\frac{5}{2}&0\\0&0&0&\boxed{1}\end{bmatrix}$$

so the system is inconsistent and the pre-image is the empty set. For the second pre-image the same procedure leads to an augmented matrix with a different vector of constants

$$\begin{bmatrix}2&-1&5&4\\-4&2&-10&-8\end{bmatrix}\xrightarrow{\text{RREF}}\begin{bmatrix}\boxed{1}&-\frac{1}{2}&\frac{5}{2}&2\\0&0&0&0\end{bmatrix}$$

This system is consistent and has infinitely many solutions, as we can see from the presence of the two free variables (x_2 and x_3) both to zero. We apply Theorem VFSLS to obtain

$$T^{-1}\left(\begin{bmatrix}4\\-8\end{bmatrix}\right)=\left\{\begin{bmatrix}2\\0\\0\end{bmatrix}+x_2\begin{bmatrix}\frac{1}{2}\\1\\0\end{bmatrix}+x_3\begin{bmatrix}-\frac{5}{2}\\0\\1\end{bmatrix}\,\middle|\, x_2,\,x_3\in\mathbb{C}\right\}$$

C31 (Robert Beezer) For the linear transformation S compute the pre-images.

$$S\colon\mathbb{C}^3\to\mathbb{C}^3,\quad S\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right)=\begin{bmatrix}a-2b-c\\3a-b+2c\\a+b+2c\end{bmatrix}$$

$$S^{-1}\left(\begin{bmatrix}-2\\5\\3\end{bmatrix}\right)\qquad\qquad S^{-1}\left(\begin{bmatrix}-5\\5\\7\end{bmatrix}\right)$$

Solution (Robert Beezer) We work from the definition of the pre-image, Definition PI. Setting

$$S\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right)=\begin{bmatrix}-2\\5\\3\end{bmatrix}$$

we arrive at a system of three equations in three variables, with an augmented matrix that we row-reduce in a search for solutions,

$$\begin{bmatrix}1&-2&-1&-2\\3&-1&2&5\\1&1&2&3\end{bmatrix}\xrightarrow{\text{RREF}}\begin{bmatrix}\boxed{1}&0&1&0\\0&\boxed{1}&1&0\\0&0&0&\boxed{1}\end{bmatrix}$$

With a leading 1 in the last column, this system is inconsistent (Theorem RCLS), and there are no values of a, b and c that will create an element of the pre-image. So the preimage is the empty set.

We work from the definition of the pre-image, Definition PI. Setting

$$S\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right)=\begin{bmatrix}-5\\5\\7\end{bmatrix}$$

we arrive at a system of three equations in three variables, with an augmented matrix that we row-reduce in a search for solutions,

$$\begin{bmatrix}1&-2&-1&-5\\3&-1&2&5\\1&1&2&7\end{bmatrix}\xrightarrow{\text{RREF}}\begin{bmatrix}\boxed{1}&0&1&3\\0&\boxed{1}&1&4\\0&0&0&0\end{bmatrix}$$

The solution set to this system, which is also the desired pre-image, can be expressed using the vector form of the solutions (Theorem VFSLS)

$$S^{-1}\left(\begin{bmatrix}-5\\5\\7\end{bmatrix}\right)=\left\{\begin{bmatrix}3\\4\\0\end{bmatrix}+c\begin{bmatrix}-1\\-1\\1\end{bmatrix}\middle|\,c\in\mathbb{C}\right\}=\begin{bmatrix}3\\4\\0\end{bmatrix}+\left\langle\left\{\begin{bmatrix}-1\\-1\\1\end{bmatrix}\right\}\right\rangle$$

Does the final expression for this set remind you of Theorem KPI?

C40 (Chris Black) If $T\colon\mathbb{C}^2\to\mathbb{C}^2$ satisfies $T\left(\begin{bmatrix}2\\1\end{bmatrix}\right)=\begin{bmatrix}3\\4\end{bmatrix}$ and $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}-1\\2\end{bmatrix}$, find $T\left(\begin{bmatrix}4\\3\end{bmatrix}\right)$.

Solution (Chris Black) Since $\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}+2\begin{bmatrix}1\\1\end{bmatrix}$, we have

$$\begin{aligned}T\left(\begin{bmatrix}4\\3\end{bmatrix}\right)&=T\left(\begin{bmatrix}2\\1\end{bmatrix}+2\begin{bmatrix}1\\1\end{bmatrix}\right)=T\left(\begin{bmatrix}2\\1\end{bmatrix}\right)+2T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)\\&=\begin{bmatrix}3\\4\end{bmatrix}+2\begin{bmatrix}-1\\2\end{bmatrix}\qquad=\begin{bmatrix}1\\8\end{bmatrix}.\end{aligned}$$

C41 (Chris Black) If $T\colon\mathbb{C}^2\to\mathbb{C}^3$ satisfies $T\left(\begin{bmatrix}2\\3\end{bmatrix}\right)=\begin{bmatrix}2\\2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}3\\4\end{bmatrix}\right)=\begin{bmatrix}-1\\0\\2\end{bmatrix}$, find the matrix representation of T.

Solution (Chris Black) First, we need to write the standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ as linear combinations of $\begin{bmatrix}2\\3\end{bmatrix}$ and $\begin{bmatrix}3\\4\end{bmatrix}$. Starting with $\mathbf{e}_1$, we see that $\mathbf{e}_1 = -4\begin{bmatrix}2\\3\end{bmatrix} + 3\begin{bmatrix}3\\4\end{bmatrix}$, so we have

$$\begin{aligned} T(\mathbf{e}_1) &= T\left(-4\begin{bmatrix}2\\3\end{bmatrix} + 3\begin{bmatrix}3\\4\end{bmatrix}\right) = -4\,T\left(\begin{bmatrix}2\\3\end{bmatrix}\right) + 3\,T\left(\begin{bmatrix}3\\4\end{bmatrix}\right)\\ &= -4\begin{bmatrix}2\\2\\1\end{bmatrix} + 3\begin{bmatrix}-1\\0\\2\end{bmatrix} = \begin{bmatrix}-11\\-8\\2\end{bmatrix}. \end{aligned}$$

Repeating the process for $\mathbf{e}_2$, we have $\mathbf{e}_2 = 3\begin{bmatrix}2\\3\end{bmatrix} - 2\begin{bmatrix}3\\4\end{bmatrix}$, and we then see that

$$\begin{aligned} T(\mathbf{e}_2) &= T\left(3\begin{bmatrix}2\\3\end{bmatrix} - 2\begin{bmatrix}3\\4\end{bmatrix}\right) = 3\,T\left(\begin{bmatrix}2\\3\end{bmatrix}\right) - 2\,T\left(\begin{bmatrix}3\\4\end{bmatrix}\right)\\ &= 3\begin{bmatrix}2\\2\\1\end{bmatrix} - 2\begin{bmatrix}-1\\0\\2\end{bmatrix} = \begin{bmatrix}8\\6\\-1\end{bmatrix}. \end{aligned}$$

Thus, the matrix representation of T is $A_T = \begin{bmatrix}-11 & 8\\-8 & 6\\2 & -1\end{bmatrix}$.

C42 (Chris Black) Define $T\colon M_{2,2} \to \mathbb{R}$ by $T\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\right) = a + b + c - d$. Find the pre-image $T^{-1}(3)$.

Solution (Chris Black) The preimage $T^{-1}(3)$ is the set of all matrices $\begin{bmatrix}a & b\\c & d\end{bmatrix}$ so that $T\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\right) = 3$. A matrix $\begin{bmatrix}a & b\\c & d\end{bmatrix}$ is in the preimage if $a + b + c - d = 3$, i.e. $d = a + b + c - 3$. This is the set. (But the set is *not* a vector space. Why not?)

$$T^{-1}(3) = \left\{\left.\begin{bmatrix}a & b\\c & a+b+c-3\end{bmatrix}\right|\, a, b, c \in \mathbb{C}\right\}$$

C43 (Chris Black) Define $T\colon P_3 \to P_2$ by $T\left(a + bx + cx^2 + dx^3\right) = b + 2cx + 3dx^2$. Find the pre-image of $\mathbf{0}$. Does this linear transformation seem familiar?

Solution (Chris Black) The preimage $T^{-1}(0)$ is the set of all polynomials $a + bx + cx^2 + dx^3$ so that $T\left(a + bx + cx^2 + dx^3\right) = 0$. Thus, $b + 2cx + 3dx^2 = 0$, where the 0 represents the zero polynomial. In order to satisfy this equation, we must have $b = 0$, $c = 0$, and $d = 0$. Thus, $T^{-1}(0)$ is precisely the set of all constant polynomials – polynomials of degree 0. Symbolically, this is $T^{-1}(0) = \{a|\, a \in \mathbb{C}\}$.

Does this seem familiar? What other operation sends constant functions to 0?

M10 (Robert Beezer) Define two linear transformations, $T\colon \mathbb{C}^4 \to \mathbb{C}^3$ and $S\colon \mathbb{C}^3 \to \mathbb{C}^2$ by

$$S\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}x_1 - 2x_2 + 3x_3\\5x_1 + 4x_2 + 2x_3\end{bmatrix} \qquad T\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\right) = \begin{bmatrix}-x_1 + 3x_2 + x_3 + 9x_4\\2x_1 + x_3 + 7x_4\\4x_1 + 2x_2 + x_3 + 2x_4\end{bmatrix}$$

Using the proof of Theorem MLTCV compute the matrix representations of the three linear transformations T, S and $S \circ T$. Discover and comment on the relationship between these three matrices.

Solution (Robert Beezer)

$$\begin{bmatrix}1 & -2 & 3\\5 & 4 & 2\end{bmatrix}\begin{bmatrix}-1 & 3 & 1 & 9\\2 & 0 & 1 & 7\\4 & 2 & 1 & 2\end{bmatrix} = \begin{bmatrix}7 & 9 & 2 & 1\\11 & 19 & 11 & 77\end{bmatrix}$$

M60 (Robert Beezer) Suppose U and V are vector spaces and define a function $Z\colon U \to V$ by $T(\mathbf{u}) = \mathbf{0}_V$ for every $\mathbf{u} \in U$. Prove that Z is a (stupid) linear transformation. (See Exercise ILT.M60, Exercise SLT.M60, Exercise IVLT.M60.)

T20 (Robert Beezer) Use the conclusion of Theorem LTLC to motivate a new definition of a linear transformation. Then prove that your new definition is equivalent to Definition LT. (Proof Technique D and Proof Technique E might be helpful if you are not sure what you are being asked to prove here.)

Theorem SER established three properties of matrix similarity that are collectively known as the defining properties of an "equivalence relation". Exercises T30 and T31 extend this idea to linear transformations.

T30 (Robert Beezer) Suppose that $T\colon U \to V$ is a linear transformation. Say that two vectors from U, $\mathbf{x}$ and $\mathbf{y}$, are **related** exactly when $T(\mathbf{x}) = T(\mathbf{y})$ in V. Prove the three properties of an equivalence relation on U: (a) for any $\mathbf{x} \in U$, $\mathbf{x}$ is related to $\mathbf{x}$, (b) if $\mathbf{x}$ is related to $\mathbf{y}$, then $\mathbf{y}$ is related to $\mathbf{x}$, and (c) if $\mathbf{x}$ is related to $\mathbf{y}$ and $\mathbf{y}$ is related to $\mathbf{z}$, then $\mathbf{x}$ is related to $\mathbf{z}$.

T31 (Robert Beezer) Equivalence relations always create a partition of the set they are defined on, via a construction called equivalence classes. For the relation in the previous problem, the equivalence classes are the pre-images. Prove directly that the collection of pre-images partition U by showing that (a) every $\mathbf{x} \in U$ is contained in some pre-image, and that (b) any two different pre-images do not have any elements in common.

Solution (Robert Beezer) Choose $\mathbf{x} \in U$, then $T(\mathbf{x}) \in V$ and we can form $T^{-1}(T(\mathbf{x}))$. Almost trivially, $\mathbf{x} \in T^{-1}(T(\mathbf{x}))$, so every vector in U is in *some* preimage. For (b), suppose that $T^{-1}(\mathbf{v}_1)$ and $T^{-1}(\mathbf{v}_2)$ are two *different* preimages, and the vector $\mathbf{u} \in U$ is an element of both. Then $T(\mathbf{u}) = \mathbf{v}_1$ and $T(\mathbf{u}) = \mathbf{v}_2$. But because T is a function, we conclude that $\mathbf{v}_1 = \mathbf{v}_2$. It then follows that $T^{-1}(\mathbf{v}_1) = T^{-1}(\mathbf{v}_2)$, contrary to our assumption that they were different. So there cannot be a common element $\mathbf{u}$.

Section ILT
Injective Linear Transformations

C10 (Robert Beezer) Each archetype below is a linear transformation. Compute the kernel for each.

Archetype M, Archetype N, Archetype O, Archetype P, Archetype Q, Archetype R, Archetype S, Archetype T, Archetype U, Archetype V, Archetype W, Archetype X

C20 (Robert Beezer) The linear transformation $T\colon \mathbb{C}^4 \to \mathbb{C}^3$ is not injective. Find two inputs $\mathbf{x}, \mathbf{y} \in \mathbb{C}^4$ that yield the same output (that is $T(\mathbf{x}) = T(\mathbf{y})$).

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 + x_3 \\ -x_1 + 3x_2 + x_3 - x_4 \\ 3x_1 + x_2 + 2x_3 - 2x_4 \end{bmatrix}$$

Solution (Robert Beezer) A linear transformation that is not injective will have a non-trivial kernel (Theorem KILT), and this is the key to finding the desired inputs. We need one non-trivial element of the kernel, so suppose that $\mathbf{z} \in \mathbb{C}^4$ is an element of the kernel,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0} = T(\mathbf{z}) = \begin{bmatrix} 2z_1 + z_2 + z_3 \\ -z_1 + 3z_2 + z_3 - z_4 \\ 3z_1 + z_2 + 2z_3 - 2z_4 \end{bmatrix}$$

Vector equality Definition CVE leads to the homogeneous system of three equations in four variables,

$$\begin{aligned} 2z_1 + z_2 + z_3 &= 0 \\ -z_1 + 3z_2 + z_3 - z_4 &= 0 \\ 3z_1 + z_2 + 2z_3 - 2z_4 &= 0 \end{aligned}$$

The coefficient matrix of this system row-reduces as

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ -1 & 3 & 1 & -1 \\ 3 & 1 & 2 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & -3 \end{bmatrix}$$

From this we can find a solution (we only need one), that is an element of $\mathcal{K}(T)$,

$$\mathbf{z} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ 1 \end{bmatrix}$$

Now, we choose a vector $\mathbf{x}$ at random and set $\mathbf{y} = \mathbf{x} + \mathbf{z}$,

$$\mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ -2 \end{bmatrix} \qquad \mathbf{y} = \mathbf{x} + \mathbf{z} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 7 \\ -1 \end{bmatrix}$$

and you can check that

$$T(\mathbf{x}) = \begin{bmatrix} 11 \\ 13 \\ 21 \end{bmatrix} = T(\mathbf{y})$$

A quicker solution is to take two elements of the kernel (in this case, scalar multiples of $\mathbf{z}$) which both get sent to $\mathbf{0}$ by T. Quicker yet, take $\mathbf{0}$ and $\mathbf{z}$ as $\mathbf{x}$ and $\mathbf{y}$, which also both get sent to $\mathbf{0}$ by T.

C25 (Robert Beezer) Define the linear transformation

$$T\colon \mathbb{C}^3 \to \mathbb{C}^2, \quad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 + 5x_3 \\ -4x_1 + 2x_2 - 10x_3 \end{bmatrix}$$

Find a basis for the kernel of T, $\mathcal{K}(T)$. Is T injective?

Solution (Robert Beezer) To find the kernel, we require all $\mathbf{x} \in \mathbb{C}^3$ such that $T(\mathbf{x}) = \mathbf{0}$. This condition is

$$\begin{bmatrix} 2x_1 - x_2 + 5x_3 \\ -4x_1 + 2x_2 - 10x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This leads to a homogeneous system of two linear equations in three variables, whose coefficient matrix row-reduces to

$$\begin{bmatrix} \boxed{1} & -\frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

With two free variables Theorem BNS yields the basis for the null space

$$\left\{ \begin{bmatrix} -\frac{5}{2} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \right\}$$

With $n(T) \neq 0$, $\mathcal{K}(T) \neq \{\mathbf{0}\}$, so Theorem KILT says T is not injective.

C26 (Chris Black) Let $A = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 \\ 2 & -1 & 1 & 0 & 1 \\ 1 & 2 & -1 & -2 & 1 \\ 1 & 3 & 2 & 1 & 2 \end{bmatrix}$ and let $T\colon \mathbb{C}^5 \to \mathbb{C}^4$ be given by $T(\mathbf{x}) = A\mathbf{x}$. Is T injective? (Hint: No calculation is required.)

Solution (Chris Black) By Theorem ILTD, if a linear transformation $T\colon U \to V$ is injective, then $\dim(U) \leq \dim(V)$. In this case, $T\colon \mathbb{C}^5 \to \mathbb{C}^4$, and $5 = \dim\left(\mathbb{C}^5\right) > \dim\left(\mathbb{C}^4\right) = 4$. Thus, T cannot possibly be injective.

C27 (Chris Black) Let $T\colon \mathbb{C}^3 \to \mathbb{C}^3$ be given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y + z \\ x - y + 2z \\ x + 2y - z \end{bmatrix}$. Find $\mathcal{K}(T)$. Is T injective?

Solution (Chris Black) If $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \mathbf{0}$, then $\begin{bmatrix} 2x + y + z \\ x - y + 2z \\ x + 2y - z \end{bmatrix} = \mathbf{0}$. Thus, we have the system

$$\begin{aligned} 2x + y + z &= 0 \\ x - y + 2z &= 0 \\ x + 2y - z &= 0 \end{aligned}$$

Thus, we are looking for the nullspace of the matrix $A_T = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$. Since A_T row-reduces to $\begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{bmatrix}$, the kernel of T is all vectors where $x = -z$ and $y = z$. Thus, $\mathcal{K}(T) = \left\langle \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\} \right\rangle$.

Since the kernel is not trivial, Theorem KILT tells us that T is not injective.

C28 (Chris Black) Let $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & -1 & 1 & 0 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 2 & 1 \end{bmatrix}$ and let $T\colon \mathbb{C}^4 \to \mathbb{C}^4$ be given by $T(\mathbf{x}) = A\mathbf{x}$. Find $\mathcal{K}(T)$. Is T injective?

Solution (Chris Black) Since T is given by matrix multiplication, $\mathcal{K}(T) = \mathcal{N}(A)$. We have

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & -1 & 1 & 0 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 2 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

The nullspace of A is $\{\mathbf{0}\}$, so the kernel of T is also trivial: $\mathcal{K}(T) = \{\mathbf{0}\}$.

C29 (Chris Black) Let $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 1 \end{bmatrix}$ and let $T\colon \mathbb{C}^4 \to \mathbb{C}^4$ be given by $T(\mathbf{x}) = A\mathbf{x}$. Find $\mathcal{K}(T)$. Is T injective?

Solution (Chris Black) Since T is given by matrix multiplication, $\mathcal{K}(T) = \mathcal{N}(A)$. We have

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1/3 & 0 \\ 0 & \boxed{1} & 1/3 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, a basis for the nullspace of A is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 3 \\ 0 \end{bmatrix} \right\}$, and the kernel is $\mathcal{K}(T) = \left\langle \left\{ \begin{bmatrix} -1 \\ -1 \\ 3 \\ 0 \end{bmatrix} \right\} \right\rangle$. Since the kernel is nontrivial, this linear transformation is not injective.

C30 (Chris Black) Let $T : M_{2,2} \to P_2$ be given by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b) + (a+c)x + (a+d)x^2$. Is T injective? Find $\mathcal{K}(T)$.

Solution (Chris Black) We can see without computing that T is not injective, since the degree of $M_{2,2}$ is larger than the degree of P_2. However, that doesn't address the question of the kernel of T. We need to find all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so that $(a+b) + (a+c)x + (a+d)x^2 = 0$. This means $a + b = 0$, $a + c = 0$, and $a + d = 0$, or equivalently, $b = d = c = -a$. Thus, the kernel is a one-dimensional subspace of $M_{2,2}$ spanned by $\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$. Symbolically, we have $\mathcal{K}(T) = \left\langle \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \right\rangle$.

C31 (Chris Black) Given that the linear transformation $T\colon \mathbb{C}^3 \to \mathbb{C}^3$, $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x+y \\ 2y+z \\ x+2z \end{bmatrix}$ is injective, show directly that $\{T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)\}$ is a linearly independent set.

Solution (Chris Black) We have

$$T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \qquad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad T(\mathbf{e}_3) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Let's put these vectors into a matrix and row reduce to test their linear independence.

$$\begin{bmatrix}2 & 1 & 0\\0 & 2 & 1\\1 & 0 & 2\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & 0\\0 & \boxed{1} & 0\\0 & 0 & \boxed{1}\end{bmatrix}$$

so the set of vectors $\{T(\mathbf{e}_1), T(\mathbf{e}_1), T(\mathbf{e}_1)\}$ is linearly independent.

C32 (Chris Black) Given that the linear transformation $T\colon \mathbb{C}^2 \to \mathbb{C}^3$, $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x+y\\2x+y\\x+2y\end{bmatrix}$ is injective, show directly that $\{T(\mathbf{e}_1), T(\mathbf{e}_2)\}$ is a linearly independent set.

Solution (Chris Black) We have $T(\mathbf{e}_1) = \begin{bmatrix}1\\2\\1\end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix}1\\1\\2\end{bmatrix}$. Putting these into a matrix as columns and row-reducing, we have

$$\begin{bmatrix}1 & 1\\2 & 1\\1 & 2\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0\\0 & \boxed{1}\\0 & 0\end{bmatrix}$$

Thus, the set of vectors $\{T(\mathbf{e}_1), T(\mathbf{e}_2)\}$ is linearly independent.

C33 (Chris Black) Given that the linear transformation $T\colon \mathbb{C}^3 \to \mathbb{C}^5$, $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}1 & 3 & 2\\0 & 1 & 1\\1 & 2 & 1\\1 & 0 & 1\\3 & 1 & 2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$ is injective, show directly that $\{T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)\}$ is a linearly independent set.

Solution (Chris Black) We have

$$T(\mathbf{e}_1) = \begin{bmatrix}1\\0\\1\\1\\3\end{bmatrix} \qquad T(\mathbf{e}_2) = \begin{bmatrix}3\\1\\2\\0\\1\end{bmatrix} \qquad T(\mathbf{e}_3) = \begin{bmatrix}2\\1\\1\\1\\2\end{bmatrix}$$

Let's row reduce the matrix of T to test linear independence.

$$\begin{bmatrix}1 & 3 & 2\\0 & 1 & 1\\1 & 2 & 1\\1 & 0 & 1\\3 & 1 & 2\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & 0\\0 & \boxed{1} & 0\\0 & 0 & \boxed{1}\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

so the set of vectors $\{T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)\}$ is linearly independent.

C40 (Robert Beezer) Show that the linear transformation R is not injective by finding two different elements of the domain, $\mathbf{x}$ and $\mathbf{y}$, such that $R(\mathbf{x}) = R(\mathbf{y})$. ($S_{22}$ is the vector space of symmetric 2×2 matrices.)

$$R\colon S_{22} \to P_1 \quad R\left(\begin{bmatrix}a & b\\b & c\end{bmatrix}\right) = (2a - b + c) + (a + b + 2c)x$$

Solution (Robert Beezer) We choose $\mathbf{x}$ to be any vector we like. A particularly cocky choice would be to choose $\mathbf{x} = \mathbf{0}$, but we will instead choose

$$\mathbf{x} = \begin{bmatrix}2 & -1\\-1 & 4\end{bmatrix}$$

Then $R(\mathbf{x}) = 9 + 9x$. Now compute the kernel of R, which by Theorem KILT we expect to be nontrivial. Setting $R\left(\begin{bmatrix}a & b\\b & c\end{bmatrix}\right)$ equal to the zero vector, $\mathbf{0} = 0 + 0x$, and equating coefficients leads to a homogeneous system of equations. Row-reducing the coefficient matrix of this system will allow us to determine the values of a, b and c that create elements of the null space of R,

$$\begin{bmatrix}2 & -1 & 1\\1 & 1 & 2\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & 1\\0 & \boxed{1} & 1\end{bmatrix}$$

We only need a single element of the null space of this coefficient matrix, so we will not compute a precise description of the whole null space. Instead, choose the free variable $c = 2$. Then

$$\mathbf{z} = \begin{bmatrix} -2 & -2 \\ -2 & 2 \end{bmatrix}$$

is the corresponding element of the kernel. We compute the desired $\mathbf{y}$ as

$$\mathbf{y} = \mathbf{x} + \mathbf{z} = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} -2 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ -3 & 6 \end{bmatrix}$$

Then check that $R(\mathbf{y}) = 9 + 9x$.

M60 (Robert Beezer) Suppose U and V are vector spaces. Define the function $Z\colon U \to V$ by $T(\mathbf{u}) = \mathbf{0}_V$ for every $\mathbf{u} \in U$. Then by Exercise LT.M60, Z is a linear transformation. Formulate a condition on U that is equivalent to Z being an injective linear transformation. In other words, fill in the blank to complete the following statement (and then give a proof): Z is injective if and only if U is . (See Exercise SLT.M60, Exercise IVLT.M60.)

T10 (Robert Beezer) Suppose $T\colon U \to V$ is a linear transformation. For which vectors $\mathbf{v} \in V$ is $T^{-1}(\mathbf{v})$ a subspace of U?

T15 (Robert Beezer) Suppose that that $T\colon U \to V$ and $S\colon V \to W$ are linear transformations. Prove the following relationship between kernels.

$$\mathcal{K}(T) \subseteq \mathcal{K}(S \circ T)$$

Solution (Robert Beezer) We are asked to prove that $\mathcal{K}(T)$ is a subset of $\mathcal{K}(S \circ T)$. Employing Definition SSET, choose $\mathbf{x} \in \mathcal{K}(T)$. Then we know that $T(\mathbf{x}) = \mathbf{0}$. So

$$\begin{aligned} (S \circ T)(\mathbf{x}) &= S(T(\mathbf{x})) && \text{Definition LTC} \\ &= S(\mathbf{0}) && \mathbf{x} \in \mathcal{K}(T) \\ &= \mathbf{0} && \text{Theorem LTTZZ} \end{aligned}$$

This qualifies $\mathbf{x}$ for membership in $\mathcal{K}(S \circ T)$.

T20 (Andy Zimmer) Suppose that A is an $m \times n$ matrix. Define the linear transformation T by

$$T\colon \mathbb{C}^n \to \mathbb{C}^m, \quad T(\mathbf{x}) = A\mathbf{x}$$

Prove that the kernel of T equals the null space of A, $\mathcal{K}(T) = \mathcal{N}(A)$.

Solution (Andy Zimmer) This is an equality of sets, so we want to establish two subset conditions (Definition SE).

First, show $\mathcal{N}(A) \subseteq \mathcal{K}(T)$. Choose $\mathbf{x} \in \mathcal{N}(A)$. Check to see if $\mathbf{x} \in \mathcal{K}(T)$,

$$\begin{aligned} T(\mathbf{x}) &= A\mathbf{x} && \text{Definition of } T \\ &= \mathbf{0} && \mathbf{x} \in \mathcal{N}(A) \end{aligned}$$

So by Definition KLT, $\mathbf{x} \in \mathcal{K}(T)$ and thus $\mathcal{N}(A) \subseteq \mathcal{K}(T)$.

Now, show $\mathcal{K}(T) \subseteq \mathcal{N}(A)$. Choose $\mathbf{x} \in \mathcal{K}(T)$. Check to see if $\mathbf{x} \in \mathcal{N}(A)$,

$$\begin{aligned} A\mathbf{x} &= T(\mathbf{x}) && \text{Definition of } T \\ &= \mathbf{0} && \mathbf{x} \in \mathcal{K}(T) \end{aligned}$$

So by Definition NSM, $\mathbf{x} \in \mathcal{N}(A)$ and thus $\mathcal{K}(T) \subseteq \mathcal{N}(A)$.

Section SLT
Surjective Linear Transformations

C10 (Robert Beezer) Each archetype below is a linear transformation. Compute the range for each.

Archetype M, Archetype N, Archetype O, Archetype P, Archetype Q, Archetype R, Archetype S, Archetype T, Archetype U, Archetype V, Archetype W, Archetype X

C20 (Robert Beezer) Example SAR concludes with an expression for a vector $\mathbf{u} \in \mathbb{C}^5$ that we believe will create the vector $\mathbf{v} \in \mathbb{C}^5$ when used to evaluate T. That is, $T(\mathbf{u}) = \mathbf{v}$. Verify this assertion by actually evaluating T with $\mathbf{u}$. If you don't have the patience to push around all these symbols, try choosing a numerical instance of $\mathbf{v}$, compute $\mathbf{u}$, and then compute $T(\mathbf{u})$, which should result in $\mathbf{v}$.

C22 (Robert Beezer) The linear transformation $S\colon \mathbb{C}^4 \to \mathbb{C}^3$ is not surjective. Find an output $\mathbf{w} \in \mathbb{C}^3$ that has an empty pre-image (that is $S^{-1}(\mathbf{w}) = \emptyset$.)

$$S\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\right) = \begin{bmatrix}2x_1 + x_2 + 3x_3 - 4x_4\\x_1 + 3x_2 + 4x_3 + 3x_4\\-x_1 + 2x_2 + x_3 + 7x_4\end{bmatrix}$$

Solution (Robert Beezer) To find an element of $\mathbb{C}^3$ with an empty pre-image, we will compute the range of the linear transformation $\mathcal{R}(S)$ and then find an element outside of this set.

By Theorem SSRLT we can evaluate S with the elements of a spanning set of the domain and create a spanning set for the range.

$$S\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\\-1\end{bmatrix} \qquad S\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\3\\2\end{bmatrix} \qquad S\left(\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}3\\4\\1\end{bmatrix} \qquad S\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-4\\3\\7\end{bmatrix}$$

So

$$\mathcal{R}(S) = \left\langle\left\{\begin{bmatrix}2\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\3\\2\end{bmatrix}, \begin{bmatrix}3\\4\\1\end{bmatrix}, \begin{bmatrix}-4\\3\\7\end{bmatrix}\right\}\right\rangle$$

This spanning set is obviously linearly dependent, so we can reduce it to a basis for $\mathcal{R}(S)$ using Theorem BRS, where the elements of the spanning set are placed as the rows of a matrix. The result is that

$$\mathcal{R}(S) = \left\langle\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}\right\rangle$$

Therefore, the unique vector in $\mathcal{R}(S)$ with a first slot equal to 6 and a second slot equal to 15 will be the linear combination

$$6\begin{bmatrix}1\\0\\-1\end{bmatrix} + 15\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}6\\15\\9\end{bmatrix}$$

So, any vector with first two components equal to 6 and 15, but with a third component different from 9, such as

$$\mathbf{w} = \begin{bmatrix}6\\15\\-63\end{bmatrix}$$

will not be an element of the range of S and will therefore have an empty pre-image. Another strategy on this problem is to *guess*. Almost any vector will lie outside the range of T, you have to be unlucky to randomly choose an element of the range. This is because the codomain has dimension 3, while the range is "much smaller" at a dimension of 2. You still need to check that your guess lies outside of the range, which generally will involve solving a system of equations that turns out to be inconsistent.

C23 (Chris Black) Determine whether or not the following linear transformation $T\colon \mathbb{C}^5 \to P_3$ is surjective:

$$T\left(\begin{bmatrix}a\\b\\c\\d\\e\end{bmatrix}\right) = a + (b+c)x + (c+d)x^2 + (d+e)x^3$$

Solution (Chris Black) The linear transformation T is surjective if for any $p(x) = \alpha + \beta x + \gamma x^2 + \delta x^3$, there is a vector $\mathbf{u} = \begin{bmatrix}a\\b\\c\\d\\e\end{bmatrix}$ in $\mathbb{C}^5$ so that $T(\mathbf{u}) = p(x)$. We need to be able to solve the system

$$\begin{aligned} a &= \alpha \\ b + c &= \beta \\ c + d &= \gamma \\ d + e &= \delta \end{aligned}$$

This system has an infinite number of solutions, one of which is $a = \alpha$, $b = \beta$, $c = 0$, $d = \gamma$ and $e = \delta - \gamma$, so that

$$\begin{aligned} T\left(\begin{bmatrix}\alpha\\\beta\\0\\\gamma\\\delta-\gamma\end{bmatrix}\right) &= \alpha + (\beta+0)x + (0+\gamma)x^2 + (\gamma + (\delta-\gamma))x^3 \\ &= \alpha + \beta x + \gamma x^2 + \delta x^3 \\ &= p(x). \end{aligned}$$

Thus, T is surjective, since for every vector $\mathbf{v} \in P_3$, there exists a vector $\mathbf{u} \in \mathbb{C}^5$ so that $T(\mathbf{u}) = \mathbf{v}$.

C24 (Chris Black) Determine whether or not the linear transformation $T\colon P_3 \to \mathbb{C}^5$ below is surjective:

$$T\left(a + bx + cx^2 + dx^3\right) = \begin{bmatrix}a+b\\b+c\\c+d\\a+c\\b+d\end{bmatrix}.$$

Solution (Chris Black) According to Theorem SLTD, if a linear transformation $T\colon U \to V$ is surjective, then $\dim(U) \geq \dim(V)$. In this example, $U = P_3$ has dimension 4, and $V = \mathbb{C}^5$ has dimension 5, so T cannot be surjective. (There is no way T can "expand" the domain P_3 to fill the codomain $\mathbb{C}^5$.)

C25 (Robert Beezer) Define the linear transformation

$$T\colon \mathbb{C}^3 \to \mathbb{C}^2, \quad T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}2x_1 - x_2 + 5x_3\\-4x_1 + 2x_2 - 10x_3\end{bmatrix}$$

Find a basis for the range of T, $\mathcal{R}(T)$. Is T surjective?

Solution (Robert Beezer) To find the range of T, apply T to the elements of a spanning set for $\mathbb{C}^3$ as suggested in Theorem SSRLT. We will use the standard basis vectors (Theorem SUVB).

$$\mathcal{R}(T) = \langle\{T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)\}\rangle = \left\langle\left\{\begin{bmatrix}2\\-4\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}, \begin{bmatrix}5\\-10\end{bmatrix}\right\}\right\rangle$$

Each of these vectors is a scalar multiple of the others, so we can toss two of them in reducing the spanning set to a linearly independent set (or be more careful and apply Theorem BCS on a matrix with these three vectors as columns). The result is the basis of the range,

$$\left\{\begin{bmatrix}1\\-2\end{bmatrix}\right\}$$

Since $\mathcal{R}(T)$ has dimension 1, and the codomain has dimension 2, they cannot be equal. So Theorem RSLT says T is not surjective.

C26 (Chris Black) Let $T\colon \mathbb{C}^3 \to \mathbb{C}^3$ be given by $T\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = \begin{bmatrix}a+b+2c\\2c\\a+b+c\end{bmatrix}$. Find a basis of $\mathcal{R}(T)$. Is T surjective?

Solution (Chris Black) The range of T is

$$\begin{aligned}\mathcal{R}(T) &= \left\{\left.\begin{bmatrix}a+b+2c\\2c\\a+b+c\end{bmatrix}\right| a,b,c\in\mathbb{C}\right\}\\ &= \left\{\left. a\begin{bmatrix}1\\0\\1\end{bmatrix} + b\begin{bmatrix}1\\0\\1\end{bmatrix} + c\begin{bmatrix}2\\2\\1\end{bmatrix}\right| a,b,c\in\mathbb{C}\right\}\\ &= \left\langle\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\2\\1\end{bmatrix}\right\}\right\rangle\end{aligned}$$

Since the vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}2\\2\\1\end{bmatrix}$ are linearly independent (why?), a basis of $\mathcal{R}(T)$ is $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\2\\1\end{bmatrix}\right\}$. Since the dimension of the range is 2 and the dimension of the codomain is 3, T is not surjective.

C27 (Chris Black) Let $T\colon \mathbb{C}^3 \to \mathbb{C}^4$ be given by $T\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = \begin{bmatrix}a+b-c\\a-b+c\\-a+b+c\\a+b+c\end{bmatrix}$. Find a basis of $\mathcal{R}(T)$. Is T surjective?

Solution (Chris Black) The range of T is

$$\begin{aligned}\mathcal{R}(T) &= \left\{\left.\begin{bmatrix}a+b-c\\a-b+c\\-a+b+c\\a+b+c\end{bmatrix}\right| a,b,c\in\mathbb{C}\right\}\\ &= \left\{\left. a\begin{bmatrix}1\\1\\-1\\1\end{bmatrix} + b\begin{bmatrix}1\\-1\\1\\1\end{bmatrix} + c\begin{bmatrix}-1\\1\\1\\1\end{bmatrix}\right| a,b,c\in\mathbb{C}\right\}\\ &= \left\langle\left\{\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\\1\end{bmatrix}\right\}\right\rangle\end{aligned}$$

By row reduction (not shown), we can see that the set

$$\left\{\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\\1\end{bmatrix}\right\}$$

are linearly independent, so is a basis of $\mathcal{R}(T)$. Since the dimension of the range is 3 and the dimension of the codomain is 4, T is not surjective. (We should have anticipated that T was not surjective since the dimension of the domain is smaller than the dimension of the codomain.)

C28 (Chris Black) Let $T\colon \mathbb{C}^4 \to M_{2,2}$ be given by $T\left(\begin{bmatrix}a\\b\\c\\d\end{bmatrix}\right) = \begin{bmatrix}a+b & a+b+c\\a+b+c & a+d\end{bmatrix}$. Find a basis of $\mathcal{R}(T)$. Is T surjective?

Solution (Chris Black) The range of T is

$$\mathcal{R}(T) = \left\{\left.\begin{bmatrix}a+b & a+b+c\\a+b+c & a+d\end{bmatrix}\right| a,b,c,d\in\mathbb{C}\right\}$$

$$= \left\{ a\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} + b\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} + c\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + d\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} \middle|\, a, b, c, d \in \mathbb{C} \right\}$$
$$= \left\langle \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} \right\rangle$$
$$= \left\langle \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} \right\rangle.$$

Can you explain the last equality above?

These three matrices are linearly independent, so a basis of $\mathcal{R}(T)$ is $\left\{\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$. Thus, T is not surjective, since the range has dimension 3 which is shy of $\dim(M_{2,2}) = 4$. (Notice that the range is actually the subspace of symmetric 2×2 matrices in $M_{2,2}$.)

C29 (Chris Black) Let $T\colon P_2 \to P_4$ be given by $T(p(x)) = x^2 p(x)$. Find a basis of $\mathcal{R}(T)$. Is T surjective?

Solution (Chris Black) If we transform the basis of P_2, then Theorem SSRLT guarantees we will have a spanning set of $\mathcal{R}(T)$. A basis of P_2 is $\{1, x, x^2\}$. If we transform the elements of this set, we get the set $\{x^2, x^3, x^4\}$ which is a spanning set for $\mathcal{R}(T)$. These three vectors are linearly independent, so $\{x^2, x^3, x^4\}$ is a basis of $\mathcal{R}(T)$.

Since $\mathcal{R}(T)$ has dimension 3, and the codomain has dimension 5, they cannot be equal. So Theorem RSLT says T is not surjective.

C30 (Chris Black) Let $T\colon P_4 \to P_3$ be given by $T(p(x)) = p'(x)$, where $p'(x)$ is the derivative. Find a basis of $\mathcal{R}(T)$. Is T surjective?

Solution (Chris Black) If we transform the basis of P_4, then Theorem SSRLT guarantees we will have a spanning set of $\mathcal{R}(T)$. A basis of P_4 is $\{1, x, x^2, x^3, x^4\}$. If we transform the elements of this set, we get the set $\{0, 1, 2x, 3x^2, 4x^3\}$ which is a spanning set for $\mathcal{R}(T)$. Reducing this to a linearly independent set, we find that $\{1, 2x, 3x^2, 4x^3\}$ is a basis of $\mathcal{R}(T)$. Since $\mathcal{R}(T)$ and P_3 both have dimension 4, T is surjective.

C40 (Robert Beezer) Show that the linear transformation T is not surjective by finding an element of the codomain, $\mathbf{v}$, such that there is no vector $\mathbf{u}$ with $T(\mathbf{u}) = \mathbf{v}$.

$$T\colon \mathbb{C}^3 \to \mathbb{C}^3, \quad T\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = \begin{bmatrix}2a + 3b - c\\2b - 2c\\a - b + 2c\end{bmatrix}$$

Solution (Robert Beezer) We wish to find an output vector $\mathbf{v}$ that has no associated input. This is the same as requiring that there is no solution to the equality

$$\mathbf{v} = T\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = \begin{bmatrix}2a + 3b - c\\2b - 2c\\a - b + 2c\end{bmatrix} = a\begin{bmatrix}2\\0\\1\end{bmatrix} + b\begin{bmatrix}3\\2\\-1\end{bmatrix} + c\begin{bmatrix}-1\\-2\\2\end{bmatrix}$$

In other words, we would like to find an element of $\mathbb{C}^3$ not in the set

$$Y = \left\langle\left\{\begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}3\\2\\-1\end{bmatrix}, \begin{bmatrix}-1\\-2\\2\end{bmatrix}\right\}\right\rangle$$

If we make these vectors the rows of a matrix, and row-reduce, Theorem BRS provides an alternate description of Y,

$$Y = \left\langle\left\{\begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}0\\4\\-5\end{bmatrix}\right\}\right\rangle$$

If we add these vectors together, and then change the third component of the result, we will create a vector that lies outside of Y, say $\mathbf{v} = \begin{bmatrix}2\\4\\9\end{bmatrix}$.

M60 (Robert Beezer) Suppose U and V are vector spaces. Define the function $Z\colon U \to V$ by $T(\mathbf{u}) = \mathbf{0}_V$ for every $\mathbf{u} \in U$. Then by Exercise LT.M60, Z is a linear transformation. Formulate a condition on V that is equivalent to Z being an surjective linear transformation. In other words, fill in the blank to complete the following statement (and then give a proof): Z is surjective if and only if V is . (See Exercise ILT.M60, Exercise IVLT.M60.)

T15 (Robert Beezer) Suppose that $T\colon U \to V$ and $S\colon V \to W$ are linear transformations. Prove the following relationship between ranges.

$$\mathcal{R}(S \circ T) \subseteq \mathcal{R}(S)$$

Solution (Robert Beezer) This question asks us to establish that one set ($\mathcal{R}(S \circ T)$) is a subset of another ($\mathcal{R}(S)$). Choose an element in the "smaller" set, say $\mathbf{w} \in \mathcal{R}(S \circ T)$. Then we know that there is a vector $\mathbf{u} \in U$ such that

$$\mathbf{w} = (S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$

Now define $\mathbf{v} = T(\mathbf{u})$, so that then

$$S(\mathbf{v}) = S(T(\mathbf{u})) = \mathbf{w}$$

This statement is sufficient to show that $\mathbf{w} \in \mathcal{R}(S)$, so $\mathbf{w}$ is an element of the "larger" set, and $\mathcal{R}(S \circ T) \subseteq \mathcal{R}(S)$.

T20 (Andy Zimmer) Suppose that A is an $m \times n$ matrix. Define the linear transformation T by

$$T\colon \mathbb{C}^n \to \mathbb{C}^m, \quad T(\mathbf{x}) = A\mathbf{x}$$

Prove that the range of T equals the column space of A, $\mathcal{R}(T) = \mathcal{C}(A)$.

Solution (Andy Zimmer) This is an equality of sets, so we want to establish two subset conditions (Definition SE).

First, show $\mathcal{C}(A) \subseteq \mathcal{R}(T)$. Choose $\mathbf{y} \in \mathcal{C}(A)$. Then by Definition CSM and Definition MVP there is a vector $\mathbf{x} \in \mathbb{C}^n$ such that $A\mathbf{x} = \mathbf{y}$. Then

$$\begin{aligned} T(\mathbf{x}) &= A\mathbf{x} && \text{Definition of } T \\ &= \mathbf{y} \end{aligned}$$

This statement qualifies $\mathbf{y}$ as a member of $\mathcal{R}(T)$ (Definition RLT), so $\mathcal{C}(A) \subseteq \mathcal{R}(T)$.

Now, show $\mathcal{R}(T) \subseteq \mathcal{C}(A)$. Choose $\mathbf{y} \in \mathcal{R}(T)$. Then by Definition RLT, there is a vector $\mathbf{x}$ in $\mathbb{C}^n$ such that $T(\mathbf{x}) = \mathbf{y}$. Then

$$\begin{aligned} A\mathbf{x} &= T(\mathbf{x}) && \text{Definition of } T \\ &= \mathbf{y} \end{aligned}$$

So by Definition CSM and Definition MVP, $\mathbf{y}$ qualifies for membership in $\mathcal{C}(A)$ and so $\mathcal{R}(T) \subseteq \mathcal{C}(A)$.

Section IVLT
Invertible Linear Transformations

C10 (Robert Beezer) The archetypes below are linear transformations of the form $T\colon U \to V$ that are invertible. For each, the inverse linear transformation is given explicitly as part of the archetype's description. Verify for each linear transformation that

$$T^{-1} \circ T = I_U \qquad\qquad T \circ T^{-1} = I_V$$

Archetype R, Archetype V, Archetype W

C20 (Robert Beezer) Determine if the linear transformation $T\colon P_2 \to M_{22}$ is (a) injective, (b) surjective, (c) invertible.

$$T\left(a + bx + cx^2\right) = \begin{bmatrix} a + 2b - 2c & 2a + 2b \\ -a + b - 4c & 3a + 2b + 2c \end{bmatrix}$$

Solution (Robert Beezer) (a) We will compute the kernel of T. Suppose that $a+bx+cx^2 \in \mathcal{K}(T)$. Then

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = T\left(a+bx+cx^2\right) = \begin{bmatrix} a+2b-2c & 2a+2b \\ -a+b-4c & 3a+2b+2c \end{bmatrix}$$

and matrix equality (Theorem ME) yields the homogeneous system of four equations in three variables,

$$\begin{aligned} a+2b-2c &= 0 \\ 2a+2b &= 0 \\ -a+b-4c &= 0 \\ 3a+2b+2c &= 0 \end{aligned}$$

The coefficient matrix of this system row-reduces as

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 2 & 0 \\ -1 & 1 & -4 \\ 3 & 2 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From the existence of non-trivial solutions to this system, we can infer non-zero polynomials in $\mathcal{K}(T)$. By Theorem KILT we then know that T is not injective.

(b) Since $3 = \dim(P_2) < \dim(M_{22}) = 4$, by Theorem SLTD T is not surjective.

(c) Since T is not surjective, it is not invertible by Theorem ILTIS.

C21 (Robert Beezer) Determine if the linear transformation $S\colon P_3 \to M_{22}$ is (a) injective, (b) surjective, (c) invertible.

$$S\left(a+bx+cx^2+dx^3\right) = \begin{bmatrix} -a+4b+c+2d & 4a-b+6c-d \\ a+5b-2c+2d & a+2c+5d \end{bmatrix}$$

Solution (Robert Beezer) (a) To check injectivity, we compute the kernel of S. To this end, suppose that $a+bx+cx^2+dx^3 \in \mathcal{K}(S)$, so

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = S\left(a+bx+cx^2+dx^3\right) = \begin{bmatrix} -a+4b+c+2d & 4a-b+6c-d \\ a+5b-2c+2d & a+2c+5d \end{bmatrix}$$

this creates the homogeneous system of four equations in four variables,

$$\begin{aligned} -a+4b+c+2d &= 0 \\ 4a-b+6c-d &= 0 \\ a+5b-2c+2d &= 0 \\ a+2c+5d &= 0 \end{aligned}$$

The coefficient matrix of this system row-reduces as,

$$\begin{bmatrix} -1 & 4 & 1 & 2 \\ 4 & -1 & 6 & -1 \\ 1 & 5 & -2 & 2 \\ 1 & 0 & 2 & 5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

We recognize the coefficient matrix as being nonsingular, so the only solution to the system is $a=b=c=d=0$, and the kernel of S is trivial, $\mathcal{K}(S) = \left\{0+0x+0x^2+0x^3\right\}$. By Theorem KILT, we see that S is injective.

(b) We can establish that S is surjective by considering the rank and nullity of S.

$$\begin{aligned} r(S) &= \dim(P_3) - n(S) && \text{Theorem RPNDD} \\ &= 4-0 \\ &= \dim(M_{22}) \end{aligned}$$

So, $\mathcal{R}(S)$ is a subspace of M_{22} (Theorem RLTS) whose dimension equals that of M_{22}. By Theorem EDYES, we gain the set equality $\mathcal{R}(S) = M_{22}$. Theorem RSLT then implies that S is surjective.

(c) Since S is both injective and surjective, Theorem ILTIS says S is invertible.

C25 (Chris Black) For each linear transformation below: (a) Find the matrix representation of T, (b) Calculate $n(T)$, (c) Calculate $r(T)$, (d) Graph the image in either $\mathbb{R}^2$ or $\mathbb{R}^3$ as appropriate, (e) How many dimensions are lost?, and (f) How many dimensions are preserved?

1. $T\colon \mathbb{C}^3 \to \mathbb{C}^3$ given by $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x\\x\\x\end{bmatrix}$

2. $T\colon \mathbb{C}^3 \to \mathbb{C}^3$ given by $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x\\y\\0\end{bmatrix}$

3. $T\colon \mathbb{C}^3 \to \mathbb{C}^2$ given by $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x\\x\end{bmatrix}$

4. $T\colon \mathbb{C}^3 \to \mathbb{C}^2$ given by $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}x\\y\end{bmatrix}$

5. $T\colon \mathbb{C}^2 \to \mathbb{C}^3$ given by $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x\\y\\0\end{bmatrix}$

6. $T\colon \mathbb{C}^2 \to \mathbb{C}^3$ given by $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x\\y\\x+y\end{bmatrix}$

C50 (Robert Beezer) Consider the linear transformation $S\colon M_{12} \to P_1$ from the set of 1×2 matrices to the set of polynomials of degree at most 1, defined by

$$S\left(\begin{bmatrix}a & b\end{bmatrix}\right) = (3a + b) + (5a + 2b)x$$

Prove that S is invertible. Then show that the linear transformation

$$R\colon P_1 \to M_{12}, \quad R\left(r + sx\right) = \begin{bmatrix}(2r - s) & (-5r + 3s)\end{bmatrix}$$

is the inverse of S, that is $S^{-1} = R$.

Solution (Robert Beezer) Determine the kernel of S first. The condition that $S\left(\begin{bmatrix}a & b\end{bmatrix}\right) = \mathbf{0}$ becomes $(3a + b) + (5a + 2b)x = 0 + 0x$. Equating coefficients of these polynomials yields the system

$$\begin{aligned} 3a + b &= 0 \\ 5a + 2b &= 0 \end{aligned}$$

This homogeneous system has a nonsingular coefficient matrix, so the only solution is $a = 0$, $b = 0$ and thus

$$\mathcal{K}(S) = \left\{\begin{bmatrix}0 & 0\end{bmatrix}\right\}$$

By Theorem KILT, we know S is injective. With $n\left(S\right) = 0$ we employ Theorem RPNDD to find

$$r\left(S\right) = r\left(S\right) + 0 = r\left(S\right) + n\left(S\right) = \dim\left(M_{12}\right) = 2 = \dim\left(P_1\right)$$

Since $\mathcal{R}(S) \subseteq P_1$ and $\dim\left(\mathcal{R}(S)\right) = \dim\left(P_1\right)$, we can apply Theorem EDYES to obtain the set equality $\mathcal{R}(S) = P_1$ and therefore S is surjective.

One of the two defining conditions of an invertible linear transformation is (Definition IVLT)

$$\begin{aligned} \left(S \circ R\right)\left(a + bx\right) &= S\left(R\left(a + bx\right)\right) \\ &= S\left(\begin{bmatrix}(2a - b) & (-5a + 3b)\end{bmatrix}\right) \\ &= \left(3(2a - b) + (-5a + 3b)\right) + \left(5(2a - b) + 2(-5a + 3b)\right)x \\ &= \left((6a - 3b) + (-5a + 3b)\right) + \left((10a - 5b) + (-10a + 6b)\right)x \\ &= a + bx \end{aligned}$$

$$= I_{P_1}(a + bx)$$

That $(R \circ S)([a \quad b]) = I_{M_{12}}([a \quad b])$ is similar.

M30 (Robert Beezer) The linear transformation S below is invertible. Find a formula for the inverse linear transformation, S^{-1}.

$$S\colon P_1 \to M_{1,2}, \quad S(a + bx) = [3a + b \quad 2a + b]$$

Solution (Robert Beezer) (Another approach to this solution would follow Example CIVLT.)

Suppose that $S^{-1}\colon M_{1,2} \to P_1$ has a form given by

$$S^{-1}(z \quad w) = (rz + sw) + (pz + qw)\,x$$

where r, s, p, q are unknown scalars. Then

$$\begin{aligned}
a + bx &= S^{-1}(S(a + bx)) \\
&= S^{-1}([3a + b \quad 2a + b]) \\
&= (r(3a + b) + s(2a + b)) + (p(3a + b) + q(2a + b))\,x \\
&= ((3r + 2s)a + (r + s)b) + ((3p + 2q)a + (p + q)b)\,x
\end{aligned}$$

Equating coefficients of these two polynomials, and then equating coefficients on a and b, gives rise to 4 equations in 4 variables,

$$\begin{aligned}
3r + 2s &= 1 \\
r + s &= 0 \\
3p + 2q &= 0 \\
p + q &= 1
\end{aligned}$$

This system has a unique solution: $r = 1$, $s = -1$, $p = -2$, $q = 3$. So the desired inverse linear transformation is

$$S^{-1}(z \quad w) = (z - w) + (-2z + 3w)\,x$$

Notice that the system of 4 equations in 4 variables could be split into two systems, each with two equations in two variables (and identical coefficient matrices). After making this split, the solution might feel like computing the inverse of a matrix (Theorem CINM). Hmmmm.

M31 (Robert Beezer) The linear transformation $R\colon M_{12} \to M_{21}$ is invertible. Determine a formula for the inverse linear transformation $R^{-1}\colon M_{21} \to M_{12}$.

$$R([a \quad b]) = \begin{bmatrix} a + 3b \\ 4a + 11b \end{bmatrix}$$

Solution (Robert Beezer) (Another approach to this solution would follow Example CIVLT.)

We are given that R is invertible. The inverse linear transformation can be formulated by considering the pre-image of a generic element of the codomain. With injectivity and surjectivity, we know that the pre-image of any element will be a set of size one — it is this lone element that will be the output of the inverse linear transformation.

Suppose that we set $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ as a generic element of the codomain, M_{21}. Then if $[r \quad s] = \mathbf{w} \in R^{-1}(\mathbf{v})$,

$$\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} &= \mathbf{v} = R(\mathbf{w}) \\
&= \begin{bmatrix} r + 3s \\ 4r + 11s \end{bmatrix}
\end{aligned}$$

So we obtain the system of two equations in the two variables r and s,

$$\begin{aligned}
r + 3s &= x \\
4r + 11s &= y
\end{aligned}$$

With a nonsingular coefficient matrix, we can solve the system using the inverse of the coefficient matrix,

$$r = -11x + 3y$$
$$s = 4x - y$$

So we define,

$$R^{-1}(\mathbf{v}) = R^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \mathbf{w} = [r \quad s] = [-11x + 3y \quad 4x - y]$$

M50 (Robert Beezer) Rework Example CIVLT, only in place of the basis B for P_2, choose instead to use the basis $C = \{1,\, 1 + x,\, 1 + x + x^2\}$. This will complicate writing a generic element of the domain of T^{-1} as a linear combination of the basis elements, and the algebra will be a bit messier, but in the end you should obtain the same formula for T^{-1}. The inverse linear transformation is what it is, and the choice of a particular basis should not influence the outcome.

M60 (Robert Beezer) Suppose U and V are vector spaces. Define the function $Z\colon U \to V$ by $T(\mathbf{u}) = \mathbf{0}_V$ for every $\mathbf{u} \in U$. Then by Exercise LT.M60, Z is a linear transformation. Formulate a condition on U and V that is equivalent to Z being an invertible linear transformation. In other words, fill in the blank to complete the following statement (and then give a proof): Z is invertible if and only if U and V are . (See Exercise ILT.M60, Exercise SLT.M60, Exercise MR.M60.)

T05 (Robert Beezer) Prove that the identity linear transformation (Definition IDLT) is both injective and surjective, and hence invertible.

T15 (Robert Beezer) Suppose that $T\colon U \to V$ is a surjective linear transformation and $\dim(U) = \dim(V)$. Prove that T is injective.

Solution (Robert Beezer) If T is surjective, then Theorem RSLT says $\mathcal{R}(T) = V$, so $r(T) = \dim(V)$ by Definition ROLT. In turn, the hypothesis gives $r(T) = \dim(U)$. Then, using Theorem RPNDD,

$$n(T) = (r(T) + n(T)) - r(T) = \dim(U) - \dim(U) = 0$$

With a null space of zero dimension, $\mathcal{K}(T) = \{\mathbf{0}\}$, and by Theorem KILT we see that T is injective. T is both injective and surjective so by Theorem ILTIS, T is invertible.

T16 (Robert Beezer) Suppose that $T\colon U \to V$ is an injective linear transformation and $\dim(U) = \dim(V)$. Prove that T is surjective.

T30 (Robert Beezer) Suppose that U and V are isomorphic vector spaces. Prove that there are infinitely many isomorphisms between U and V.

Solution (Robert Beezer) Since U and V are isomorphic, there is at least one isomorphism between them (Definition IVS), say $T\colon U \to V$. As such, T is an invertible linear transformation.

For $\alpha \in \mathbb{C}$ define the linear transformation $S\colon V \to V$ by $S(\mathbf{v}) = \alpha\mathbf{v}$. Convince yourself that when $\alpha \neq 0$, S is an invertible linear transformation (Definition IVLT). Then the composition, $S \circ T\colon U \to V$, is an invertible linear transformation by Theorem CIVLT. Once convinced that each non-zero value of α gives rise to a different functions for $S \circ T$, then we have constructed infinitely many isomorphisms from U to V.

T40 (Robert Beezer) Suppose $T\colon U \to V$ and $S\colon V \to W$ are linear transformations and $\dim(U) = \dim(V) = \dim(W)$. Suppose that $S \circ T$ is invertible. Prove that S and T are individually invertible (this could be construed as a converse of Theorem CIVLT).

Solution (Robert Beezer) Since $S \circ T$ is invertible, by Theorem ILTIS $S \circ T$ is injective and therefore has a trivial kernel by Theorem KILT. Then

$$\begin{aligned} \mathcal{K}(T) &\subseteq \mathcal{K}(S \circ T) && \text{Exercise ILT.T15} \\ &= \{\mathbf{0}\} && \text{Theorem KILT} \end{aligned}$$

Since T has a trivial kernel, by Theorem KILT, T is injective. Also,

$$r(T) = \dim(U) - n(T) \qquad \text{Theorem RPNDD}$$

$$
\begin{aligned}
&= \dim(U) - 0 && \text{Theorem NOILT}\\
&= \dim(V) && \text{Hypothesis}
\end{aligned}
$$

Since $\mathcal{R}(T) \subseteq V$, Theorem EDYES gives $\mathcal{R}(T) = V$, so by Theorem RSLT, T is surjective. Finally, by Theorem ILTIS, T is invertible.

Since $S \circ T$ is invertible, by Theorem ILTIS $S \circ T$ is surjective and therefore has a full range by Theorem RSLT. Then

$$
\begin{aligned}
W &= \mathcal{R}(S \circ T) && \text{Theorem RSLT}\\
&\subseteq \mathcal{R}(S) && \text{Exercise SLT.T15}
\end{aligned}
$$

Since $\mathcal{R}(S) \subseteq W$ we have $\mathcal{R}(S) = W$ and by Theorem RSLT, S is surjective. By an application of Theorem RPNDD similar to the first part of this solution, we see that S has a trivial kernel, is therefore injective (Theorem KILT), and thus invertible (Theorem ILTIS).

Chapter R
Representations

Section VR
Vector Representations

C10 (Robert Beezer) In the vector space $\mathbb{C}^3$, compute the vector representation $\rho_B(\mathbf{v})$ for the basis B and vector $\mathbf{v}$ below.

$$B = \left\{ \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \right\} \qquad \mathbf{v} = \begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix}$$

Solution (Robert Beezer) We need to express the vector $\mathbf{v}$ as a linear combination of the vectors in B. Theorem VRRB tells us we will be able to do this, and do it uniquely. The vector equation

$$a_1 \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix}$$

becomes (via Theorem SLSLC) a system of linear equations with augmented matrix,

$$\begin{bmatrix} 2 & 1 & 3 & 11 \\ -2 & 3 & 5 & 5 \\ 2 & 1 & 2 & 8 \end{bmatrix}$$

This system has the unique solution $a_1 = 2$, $a_2 = -2$, $a_3 = 3$. So by Definition VR,

$$\rho_B(\mathbf{v}) = \rho_B\left(\begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix} \right) = \rho_B\left(2 \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

C20 (Robert Beezer) Rework Example CM32 replacing the basis B by the basis

$$C = \left\{ \begin{bmatrix} -14 & -9 \\ 10 & 10 \\ -6 & -2 \end{bmatrix}, \begin{bmatrix} -7 & -4 \\ 5 & 5 \\ -3 & -1 \end{bmatrix}, \begin{bmatrix} -3 & -1 \\ 0 & -2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -7 & -4 \\ 3 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ -3 & -3 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & -2 \\ 1 & 1 \end{bmatrix} \right\}$$

Solution (Robert Beezer) The following computations replicate the computations given in Example CM32, only using the basis C.

$$\rho_C\left(\begin{bmatrix} 3 & 7 \\ -2 & 4 \\ 0 & -3 \end{bmatrix} \right) = \begin{bmatrix} -9 \\ 12 \\ -6 \\ 7 \\ -2 \\ -1 \end{bmatrix} \qquad \rho_C\left(\begin{bmatrix} -1 & 3 \\ 4 & 8 \\ -2 & 5 \end{bmatrix} \right) = \begin{bmatrix} -11 \\ 34 \\ -4 \\ -1 \\ 16 \\ 5 \end{bmatrix}$$

$$6 \begin{bmatrix} -9 \\ 12 \\ -6 \\ 7 \\ -2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -11 \\ 34 \\ -4 \\ -1 \\ 16 \\ 5 \end{bmatrix} = \begin{bmatrix} -76 \\ 140 \\ -44 \\ 40 \\ 20 \\ 4 \end{bmatrix} \qquad \rho_C^{-1}\left(\begin{bmatrix} -76 \\ 140 \\ -44 \\ 40 \\ 20 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 16 & 48 \\ -4 & 40 \\ -4 & -8 \end{bmatrix}$$

M10 (Andy Zimmer) Prove that the set S below is a basis for the vector space of 2×2 matrices, M_{22}. Do this by choosing a natural basis for M_{22} and coordinatizing the elements of S with respect to this basis. Examine the resulting set of column vectors from $\mathbb{C}^4$ and apply the Coordinatization Principle.

$$S = \left\{ \begin{bmatrix} 33 & 99 \\ 78 & -9 \end{bmatrix}, \begin{bmatrix} -16 & -47 \\ -36 & 2 \end{bmatrix}, \begin{bmatrix} 10 & 27 \\ 17 & 3 \end{bmatrix}, \begin{bmatrix} -2 & -7 \\ -6 & 4 \end{bmatrix} \right\}$$

M20 (Tyler Ueltschi) The set $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis of the vector space P_3, polynomials with degree 3 or less. Therefore ρ_B is a linear transformation, according to Theorem VRLT. Find a "formula" for ρ_B. In other words, find an expression for $\rho_B\left(a + bx + cx^2 + dx^3\right)$.

$$\mathbf{v}_1 = 1 - 5x - 22x^2 + 3x^3 \qquad \mathbf{v}_2 = -2 + 11x + 49x^2 - 8x^3$$
$$\mathbf{v}_3 = -1 + 7x + 33x^2 - 8x^3 \qquad \mathbf{v}_4 = -1 + 4x + 16x^2 + x^3$$

Solution (Robert Beezer) Our strategy is to determine the values of the linear transformation on a "nice" basis for the domain, and then apply the ideas of Theorem LTDB to obtain our formula. ρ_B is a linear transformation of the form $\rho_B\colon P_3 \to \mathbb{C}^4$, so for a basis of the domain we choose a very simple one: $C = \left\{1, x, x^2, x^3\right\}$. We now give the vector representations of the elements of C, which are obtained by solving the relevant systems of equations obtained from linear combinations of the elements of B.

$$\rho_B(1) = \begin{bmatrix} 20 \\ 7 \\ 1 \\ 4 \end{bmatrix} \qquad \rho_B(x) = \begin{bmatrix} 17 \\ 14 \\ -8 \\ -3 \end{bmatrix} \qquad \rho_B\left(x^2\right) = \begin{bmatrix} -3 \\ -3 \\ 2 \\ 1 \end{bmatrix} \qquad \rho_B\left(x^3\right) = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

This is enough information to determine the linear transformation uniquely, and in particular, to allow us to use Theorem LTLC to construct a formula.

$$\begin{aligned}
\rho_B\left(a + bx + cx^2 + dx^3\right) &= a\rho_B(1) + b\rho_B(x) + c\rho_B\left(x^2\right) + d\rho_B\left(x^3\right) \\
&= a\begin{bmatrix} 20 \\ 7 \\ 1 \\ 4 \end{bmatrix} + b\begin{bmatrix} 17 \\ 14 \\ -8 \\ -3 \end{bmatrix} + c\begin{bmatrix} -3 \\ -3 \\ 2 \\ 1 \end{bmatrix} + d\begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 20a + 17b - 3c \\ 7a + 14b - 3c - d \\ a - 8b + 2c + d \\ 4a - 3b + c + d \end{bmatrix}
\end{aligned}$$

Section MR
Matrix Representations

C10 (Robert Beezer) Example KVMR concludes with a basis for the kernel of the linear transformation T. Compute the value of T for each of these two basis vectors. Did you get what you expected?

C20 (Robert Beezer) Compute the matrix representation of T relative to the bases B and C.

$$T\colon P_3 \to \mathbb{C}^3, \quad T\left(a + bx + cx^2 + dx^3\right) = \begin{bmatrix} 2a - 3b + 4c - 2d \\ a + b - c + d \\ 3a + 2c - 3d \end{bmatrix}$$

$$B = \left\{1, x, x^2, x^3\right\} \qquad C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Solution (Robert Beezer) Apply Definition MR,

$$\rho_C(T(1)) = \rho_C\left(\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}\right) = \rho_C\left(1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-2)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

$$\begin{aligned}
\rho_C\left(T\left(x\right)\right) &= \rho_C\left(\begin{bmatrix}-3\\1\\0\end{bmatrix}\right) = \rho_C\left((-4)\begin{bmatrix}1\\0\\0\end{bmatrix} + 1\begin{bmatrix}1\\1\\0\end{bmatrix} + 0\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}-4\\1\\0\end{bmatrix}\\
\rho_C\left(T\left(x^2\right)\right) &= \rho_C\left(\begin{bmatrix}4\\-1\\2\end{bmatrix}\right) = \rho_C\left(5\begin{bmatrix}1\\0\\0\end{bmatrix} + (-3)\begin{bmatrix}1\\1\\0\end{bmatrix} + 2\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}5\\-3\\2\end{bmatrix}\\
\rho_C\left(T\left(x^3\right)\right) &= \rho_C\left(\begin{bmatrix}-2\\1\\-3\end{bmatrix}\right) = \rho_C\left((-3)\begin{bmatrix}1\\0\\0\end{bmatrix} + 4\begin{bmatrix}1\\1\\0\end{bmatrix} + (-3)\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\4\\-3\end{bmatrix}
\end{aligned}$$

These four vectors are the columns of the matrix representation,

$$M^T_{B,C} = \begin{bmatrix}1 & -4 & 5 & -3\\ -2 & 1 & -3 & 4\\ 3 & 0 & 2 & -3\end{bmatrix}$$

C21 (Robert Beezer) Find a matrix representation of the linear transformation T relative to the bases B and C.

$$\begin{aligned}
&T\colon P_2 \to \mathbb{C}^2, \quad T\left(p(x)\right) = \begin{bmatrix}p(1)\\p(3)\end{bmatrix}\\
&B = \left\{2 - 5x + x^2,\, 1 + x - x^2,\, x^2\right\}\\
&C = \left\{\begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}\right\}
\end{aligned}$$

Solution (Robert Beezer) Applying Definition MR,

$$\begin{aligned}
\rho_C\left(T\left(2 - 5x + x^2\right)\right) &= \rho_C\left(\begin{bmatrix}-2\\-4\end{bmatrix}\right) = \rho_C\left(2\begin{bmatrix}3\\4\end{bmatrix} + (-4)\begin{bmatrix}2\\3\end{bmatrix}\right) = \begin{bmatrix}2\\-4\end{bmatrix}\\
\rho_C\left(T\left(1 + x - x^2\right)\right) &= \rho_C\left(\begin{bmatrix}1\\-5\end{bmatrix}\right) = \rho_C\left(13\begin{bmatrix}3\\4\end{bmatrix} + (-19)\begin{bmatrix}2\\3\end{bmatrix}\right) = \begin{bmatrix}13\\-19\end{bmatrix}\\
\rho_C\left(T\left(x^2\right)\right) &= \rho_C\left(\begin{bmatrix}1\\9\end{bmatrix}\right) = \rho_C\left((-15)\begin{bmatrix}3\\4\end{bmatrix} + 23\begin{bmatrix}2\\3\end{bmatrix}\right) = \begin{bmatrix}-15\\23\end{bmatrix}
\end{aligned}$$

So the resulting matrix representation is

$$M^T_{B,C} = \begin{bmatrix}2 & 13 & -15\\ -4 & -19 & 23\end{bmatrix}$$

C22 (Robert Beezer) Let S_{22} be the vector space of 2×2 symmetric matrices. Build the matrix representation of the linear transformation $T\colon P_2 \to S_{22}$ relative to the bases B and C and then use this matrix representation to compute $T\left(3 + 5x - 2x^2\right)$.

$$B = \left\{1,\, 1 + x,\, 1 + x + x^2\right\} \qquad C = \left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$$

$$T\left(a + bx + cx^2\right) = \begin{bmatrix}2a - b + c & a + 3b - c\\ a + 3b - c & a - c\end{bmatrix}$$

Solution (Robert Beezer) Input to T the vectors of the basis B and coordinatize the outputs relative to C,

$$\begin{aligned}
\rho_C\left(T\left(1\right)\right) &= \rho_C\left(\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}\right) = \rho_C\left(2\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right) = \begin{bmatrix}2\\1\\1\end{bmatrix}\\
\rho_C\left(T\left(1 + x\right)\right) &= \rho_C\left(\begin{bmatrix}1 & 4\\4 & 1\end{bmatrix}\right) = \rho_C\left(1\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 4\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right) = \begin{bmatrix}1\\4\\1\end{bmatrix}\\
\rho_C\left(T\left(1 + x + x^2\right)\right) &= \rho_C\left(\begin{bmatrix}2 & 3\\3 & 0\end{bmatrix}\right) = \rho_C\left(2\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 3\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + 0\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right) = \begin{bmatrix}2\\3\\0\end{bmatrix}
\end{aligned}$$

Applying Definition MR we have the matrix representation

$$M^T_{B,C} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 4 & 3 \\ 1 & 1 & 0 \end{bmatrix}$$

To compute $T\left(3 + 5x - 2x^2\right)$ employ Theorem FTMR,

$$\begin{aligned} T\left(3 + 5x - 2x^2\right) &= \rho_C^{-1}\left(M^T_{B,C}\rho_B\left(3 + 5x - 2x^2\right)\right) \\ &= \rho_C^{-1}\left(M^T_{B,C}\rho_B\left((-2)(1) + 7(1+x) + (-2)(1 + x + x^2)\right)\right) \\ &= \rho_C^{-1}\left(\begin{bmatrix} 2 & 1 & 2 \\ 1 & 4 & 3 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 7 \\ -2 \end{bmatrix}\right) \\ &= \rho_C^{-1}\left(\begin{bmatrix} -1 \\ 20 \\ 5 \end{bmatrix}\right) \\ &= (-1)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 20\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 5\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 20 \\ 20 & 5 \end{bmatrix} \end{aligned}$$

You can, of course, check your answer by evaluating $T\left(3 + 5x - 2x^2\right)$ directly.

C25 (Robert Beezer) Use a matrix representation to determine if the linear transformation $T\colon P_3 \to M_{22}$ is surjective.

$$T\left(a + bx + cx^2 + dx^3\right) = \begin{bmatrix} -a + 4b + c + 2d & 4a - b + 6c - d \\ a + 5b - 2c + 2d & a + 2c + 5d \end{bmatrix}$$

Solution (Robert Beezer) Choose bases B and C for the matrix representation,

$$B = \left\{1,\, x,\, x^2,\, x^3\right\} \qquad\qquad C = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

Input to T the vectors of the basis B and coordinatize the outputs relative to C,

$$\rho_C\left(T\left(1\right)\right) = \rho_C\left(\begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix}\right) = \rho_C\left((-1)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 4\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

$$\rho_C\left(T\left(x\right)\right) = \rho_C\left(\begin{bmatrix} 4 & -1 \\ 5 & 0 \end{bmatrix}\right) = \rho_C\left(4\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 5\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -1 \\ 5 \\ 0 \end{bmatrix}$$

$$\rho_C\left(T\left(x^2\right)\right) = \rho_C\left(\begin{bmatrix} 1 & 6 \\ -2 & 2 \end{bmatrix}\right) = \rho_C\left(1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 6\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (-2)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 6 \\ -2 \\ 2 \end{bmatrix}$$

$$\rho_C\left(T\left(x^3\right)\right) = \rho_C\left(\begin{bmatrix} 2 & -1 \\ 2 & 5 \end{bmatrix}\right) = \rho_C\left(2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 5\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 5 \end{bmatrix}$$

Applying Definition MR we have the matrix representation

$$M^T_{B,C} = \begin{bmatrix} -1 & 4 & 1 & 2 \\ 4 & -1 & 6 & -1 \\ 1 & 5 & -2 & 2 \\ 1 & 0 & 2 & 5 \end{bmatrix}$$

Properties of this matrix representation will translate to properties of the linear transformation The matrix representation is nonsingular since it row-reduces to the identity matrix (Theorem NMRRI) and therefore has a column space equal to $\mathbb{C}^4$ (Theorem CNMB). The column space of the matrix representation is isomorphic to the range of the linear transformation (Theorem RCSI). So the range of T has dimension 4, equal to the dimension of the codomain M_{22}. By Theorem ROSLT, T is surjective.

C30 (Robert Beezer) Find bases for the kernel and range of the linear transformation S below.

$$S\colon M_{22} \to P_2,\quad S\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+2b+5c-4d) + (3a-b+8c+2d)x + (a+b+4c-2d)x^2$$

Solution (Robert Beezer) These subspaces will be easiest to construct by analyzing a matrix representation of S. Since we can use any matrix representation, we might as well use natural bases that allow us to construct the matrix representation quickly and easily,

$$B = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\} \qquad C = \left\{1,\, x,\, x^2\right\}$$

then we can practically build the matrix representation on sight,

$$M^S_{B,C} = \begin{bmatrix} 1 & 2 & 5 & -4 \\ 3 & -1 & 8 & 2 \\ 1 & 1 & 4 & -2 \end{bmatrix}$$

The first step is to find bases for the null space and column space of the matrix representation. Row-reducing the matrix representation we find,

$$\begin{bmatrix} \boxed{1} & 0 & 3 & 0 \\ 0 & \boxed{1} & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So by Theorem BNS and Theorem BCS, we have

$$\mathcal{N}\!\left(M^S_{B,C}\right) = \left\langle\left\{\begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix},\, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}\right\}\right\rangle \qquad \mathcal{C}\!\left(M^S_{B,C}\right) = \left\langle\left\{\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix},\, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right\}\right\rangle$$

Now, the proofs of Theorem KNSI and Theorem RCSI tell us that we can apply ρ_B^{-1} and ρ_C^{-1} (respectively) to "un-coordinatize" and get bases for the kernel and range of the linear transformation S itself,

$$\mathcal{K}(S) = \left\langle\left\{\begin{bmatrix} -3 & -1 \\ 1 & 0 \end{bmatrix},\, \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\right\}\right\rangle \qquad \mathcal{R}(S) = \left\langle\left\{1+3x+x^2,\, 2-x+x^2\right\}\right\rangle$$

C40 (Robert Beezer) Let S_{22} be the set of 2×2 symmetric matrices. Verify that the linear transformation R is invertible and find R^{-1}.

$$R\colon S_{22} \to P_2,\quad R\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = (a-b) + (2a-3b-2c)x + (a-b+c)x^2$$

Solution (Robert Beezer) The analysis of R will be easiest if we analyze a matrix representation of R. Since we can use any matrix representation, we might as well use natural bases that allow us to construct the matrix representation quickly and easily,

$$B = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\} \qquad C = \left\{1,\, x,\, x^2\right\}$$

then we can practically build the matrix representation on sight,

$$M^R_{B,C} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -3 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

This matrix representation is invertible (it has a nonzero determinant of -1, Theorem SMZD, Theorem NI) so Theorem IMR tells us that the linear transformation R is also invertible. To find a formula for R^{-1} we compute,

$$\begin{aligned} R^{-1}\left(a+bx+cx^2\right) &= \rho_B^{-1}\left(M^{R^{-1}}_{C,B}\,\rho_C\left(a+bx+cx^2\right)\right) && \text{Theorem FTMR} \\ &= \rho_B^{-1}\left(\left(M^R_{B,C}\right)^{-1}\rho_C\left(a+bx+cx^2\right)\right) && \text{Theorem IMR} \\ &= \rho_B^{-1}\left(\left(M^R_{B,C}\right)^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) && \text{Definition VR} \end{aligned}$$

$$
\begin{aligned}
&= \rho_B^{-1}\left(\begin{bmatrix} 5 & -1 & -2 \\ 4 & -1 & -2 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) && \text{Definition MI} \\
&= \rho_B^{-1}\left(\begin{bmatrix} 5a-b-2c \\ 4a-b-2c \\ -a+c \end{bmatrix}\right) && \text{Definition MVP} \\
&= \begin{bmatrix} 5a-b-2c & 4a-b-2c \\ 4a-b-2c & -a+c \end{bmatrix} && \text{Definition VR}
\end{aligned}
$$

C41 (Robert Beezer) Prove that the linear transformation S is invertible. Then find a formula for the inverse linear transformation, S^{-1}, by employing a matrix inverse.

$$S\colon P_1 \to M_{1,2}, \quad S(a+bx) = \begin{bmatrix} 3a+b & 2a+b \end{bmatrix}$$

Solution (Robert Beezer) First, build a matrix representation of S (Definition MR). We are free to choose whatever bases we wish, so we should choose ones that are easy to work with, such as

$$
\begin{aligned}
B &= \{1,\, x\} \\
C &= \{\begin{bmatrix} 1 & 0 \end{bmatrix},\, \begin{bmatrix} 0 & 1 \end{bmatrix}\}
\end{aligned}
$$

The resulting matrix representation is then

$$M^T_{B,C} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

this matrix is invertible, since it has a nonzero determinant, so by Theorem IMR the linear transformation S is invertible. We can use the matrix inverse and Theorem IMR to find a formula for the inverse linear transformation,

$$
\begin{aligned}
S^{-1}\left(\begin{bmatrix} a & b \end{bmatrix}\right) &= \rho_B^{-1}\left(M^{S^{-1}}_{C,B}\,\rho_C\left(\begin{bmatrix} a & b \end{bmatrix}\right)\right) && \text{Theorem FTMR} \\
&= \rho_B^{-1}\left(\left(M^S_{B,C}\right)^{-1}\rho_C\left(\begin{bmatrix} a & b \end{bmatrix}\right)\right) && \text{Theorem IMR} \\
&= \rho_B^{-1}\left(\left(M^S_{B,C}\right)^{-1}\begin{bmatrix} a \\ b \end{bmatrix}\right) && \text{Definition VR} \\
&= \rho_B^{-1}\left(\left(\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} a \\ b \end{bmatrix}\right) \\
&= \rho_B^{-1}\left(\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}\right) && \text{Definition MI} \\
&= \rho_B^{-1}\left(\begin{bmatrix} a-b \\ -2a+3b \end{bmatrix}\right) && \text{Definition MVP} \\
&= (a-b) + (-2a+3b)x && \text{Definition VR}
\end{aligned}
$$

C42 (Robert Beezer) The linear transformation $R\colon M_{12} \to M_{21}$ is invertible. Use a matrix representation to determine a formula for the inverse linear transformation $R^{-1}\colon M_{21} \to M_{12}$.

$$R\left(\begin{bmatrix} a & b \end{bmatrix}\right) = \begin{bmatrix} a+3b \\ 4a+11b \end{bmatrix}$$

Solution (Robert Beezer) Choose bases B and C for M_{12} and M_{21} (respectively),

$$B = \{\begin{bmatrix} 1 & 0 \end{bmatrix},\, \begin{bmatrix} 0 & 1 \end{bmatrix}\}\; C = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix},\, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

The resulting matrix representation is

$$M^R_{B,C} = \begin{bmatrix} 1 & 3 \\ 4 & 11 \end{bmatrix}$$

This matrix is invertible (its determinant is nonzero, Theorem SMZD), so by Theorem IMR, we can compute the matrix representation of R^{-1} with a matrix inverse (Theorem TTMI),

$$M^{R^{-1}}_{C,B} = \begin{bmatrix} 1 & 3 \\ 4 & 11 \end{bmatrix}^{-1} = \begin{bmatrix} -11 & 3 \\ 4 & -1 \end{bmatrix}$$

To obtain a general formula for R^{-1}, use Theorem FTMR,

$$\begin{aligned}
R^{-1}\left(\begin{bmatrix}x\\y\end{bmatrix}\right) &= \rho_B^{-1}\left(M_{C,B}^{R^{-1}}\rho_C\left(\begin{bmatrix}x\\y\end{bmatrix}\right)\right)\\
&= \rho_B^{-1}\left(\begin{bmatrix}-11 & 3\\ 4 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\right)\\
&= \rho_B^{-1}\left(\begin{bmatrix}-11x+3y\\ 4x-y\end{bmatrix}\right)\\
&= \begin{bmatrix}-11x+3y & 4x-y\end{bmatrix}
\end{aligned}$$

C50 (Robert Beezer) Use a matrix representation to find a basis for the range of the linear transformation L.

$$L\colon M_{22}\to P_2,\quad T\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right) = (a+2b+4c+d)+(3a+c-2d)x+(-a+b+3c+3d)x^2$$

Solution (Robert Beezer) As usual, build any matrix representation of L, most likely using a "nice" bases, such as

$$\begin{aligned}
B &= \left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\ \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\ \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},\ \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right\}\\
C &= \left\{1,\ x,\ x^2\right\}
\end{aligned}$$

Then the matrix representation (Definition MR) is,

$$M_{B,C}^{L} = \begin{bmatrix}1 & 2 & 4 & 1\\ 3 & 0 & 1 & -2\\ -1 & 1 & 3 & 3\end{bmatrix}$$

Theorem RCSI tells us that we can compute the column space of the matrix representation, then use the isomorphism ρ_C^{-1} to convert the column space of the matrix representation into the range of the linear transformation. So we first analyze the matrix representation,

$$\begin{bmatrix}1 & 2 & 4 & 1\\ 3 & 0 & 1 & -2\\ -1 & 1 & 3 & 3\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & 0 & -1\\ 0 & \boxed{1} & 0 & -1\\ 0 & 0 & \boxed{1} & 1\end{bmatrix}$$

With three nonzero rows in the reduced row-echelon form of the matrix, we know the column space has dimension 3. Since P_2 has dimension 3 (Theorem DP), the range must be all of P_2. So *any* basis of P_2 would suffice as a basis for the range. For instance, C itself would be a correct answer.

A more laborious approach would be to use Theorem BCS and choose the first three columns of the matrix representation as a basis for the range of the matrix representation. These could then be "un-coordinatized" with ρ_C^{-1} to yield a ("not nice") basis for P_2.

C51 (Robert Beezer) Use a matrix representation to find a basis for the kernel of the linear transformation L.

$$L\colon M_{22}\to P_2,\quad T\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right) = (a+2b+4c+d)+(3a+c-2d)x+(-a+b+3c+3d)x^2$$

C52 (Robert Beezer) Find a basis for the kernel of the linear transformation $T\colon P_2\to M_{22}$.

$$T\left(a+bx+cx^2\right) = \begin{bmatrix}a+2b-2c & 2a+2b\\ -a+b-4c & 3a+2b+2c\end{bmatrix}$$

Solution (Robert Beezer) Choose bases B and C for the matrix representation,

$$B=\left\{1,\ x,\ x^2\right\}\qquad C=\left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\ \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\ \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},\ \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right\}$$

Input to T the vectors of the basis B and coordinatize the outputs relative to C,

$$\rho_C\left(T\left(1\right)\right) = \rho_C\left(\begin{bmatrix}1 & 2\\-1 & 3\end{bmatrix}\right) = \rho_C\left(1\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 2\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + (-1)\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + 3\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right) = \begin{bmatrix}1\\2\\-1\\3\end{bmatrix}$$

$$\rho_C\left(T\left(x\right)\right) = \rho_C\left(\begin{bmatrix}2 & 2\\1 & 2\end{bmatrix}\right) = \rho_C\left(2\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 2\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + 2\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right) = \begin{bmatrix}2\\2\\1\\2\end{bmatrix}$$

$$\rho_C\left(T\left(x^2\right)\right) = \rho_C\left(\begin{bmatrix}-2 & 0\\-4 & 2\end{bmatrix}\right) = \rho_C\left((-2)\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 0\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + (-4)\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + 2\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right) = \begin{bmatrix}-2\\0\\-4\\2\end{bmatrix}$$

Applying Definition MR we have the matrix representation

$$M^T_{B,C} = \begin{bmatrix}1 & 2 & -2\\2 & 2 & 0\\-1 & 1 & -4\\3 & 2 & 2\end{bmatrix}$$

The null space of the matrix representation is isomorphic (via ρ_B) to the kernel of the linear transformation (Theorem KNSI). So we compute the null space of the matrix representation by first row-reducing the matrix to,

$$\begin{bmatrix}\boxed{1} & 0 & 2\\0 & \boxed{1} & -2\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

Employing Theorem BNS we have

$$\mathcal{N}\left(M^T_{B,C}\right) = \left\langle\left\{\begin{bmatrix}-2\\2\\1\end{bmatrix}\right\}\right\rangle$$

We only need to uncoordinatize this one basis vector to get a basis for $\mathcal{K}(T)$,

$$\mathcal{K}(T) = \left\langle\left\{\rho_B^{-1}\left(\begin{bmatrix}-2\\2\\1\end{bmatrix}\right)\right\}\right\rangle = \left\langle\left\{-2 + 2x + x^2\right\}\right\rangle$$

M20 (Robert Beezer) The linear transformation D performs differentiation on polynomials. Use a matrix representation of D to find the rank and nullity of D.

$$D\colon P_n \to P_n, \quad D\left(p(x)\right) = p'(x)$$

Solution (Robert Beezer) Build a matrix representation (Definition MR) with the set

$$B = \left\{1,\, x,\, x^2,\, \ldots,\, x^n\right\}$$

employed as a basis of both the domain and codomain. Then

$$\rho_B\left(D\left(1\right)\right) = \rho_B\left(0\right) = \begin{bmatrix}0\\0\\0\\\vdots\\0\\0\end{bmatrix} \qquad \rho_B\left(D\left(x\right)\right) = \rho_B\left(1\right) = \begin{bmatrix}1\\0\\0\\\vdots\\0\\0\end{bmatrix}$$

$$\rho_B\left(D\left(x^2\right)\right) = \rho_B\left(2x\right) = \begin{bmatrix}0\\2\\0\\\vdots\\0\\0\end{bmatrix} \qquad \rho_B\left(D\left(x^3\right)\right) = \rho_B\left(3x^2\right) = \begin{bmatrix}0\\0\\3\\\vdots\\0\\0\end{bmatrix}$$

$$\vdots$$

$$\rho_B\left(D\left(x^n\right)\right) = \rho_B\left(nx^{n-1}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ n \\ 0 \end{bmatrix}$$

and the resulting matrix representation is

$$M^D_{B,B} = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 3 & \dots & 0 & 0 \\ & \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & n \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

This $(n+1) \times (n+1)$ matrix is very close to being in reduced row-echelon form. Multiply row i by $\frac{1}{i}$, for $1 \leq i \leq n$, to convert it to reduced row-echelon form. From this we can see that matrix representation $M^D_{B,B}$ has rank n and nullity 1. Applying Theorem RCSI and Theorem KNSI tells us that the linear transformation D will have the same values for the rank and nullity, as well.

M60 (Robert Beezer) Suppose U and V are vector spaces and define a function $Z\colon U \to V$ by $T(\mathbf{u}) = \mathbf{0}_V$ for every $\mathbf{u} \in U$. Then Exercise IVLT.M60 asks you to formulate the theorem: Z is invertible if and only if $U = \{\mathbf{0}_U\}$ and $V = \{\mathbf{0}_V\}$. What would a matrix representation of Z look like in this case? How does Theorem IMR read in this case?

M80 (Robert Beezer) In light of Theorem KNSI and Theorem MRCLT, write a short comparison of Exercise MM.T40 with Exercise ILT.T15.

M81 (Robert Beezer) In light of Theorem RCSI and Theorem MRCLT, write a short comparison of Exercise CRS.T40 with Exercise SLT.T15.

M82 (Robert Beezer) In light of Theorem MRCLT and Theorem IMR, write a short comparison of Theorem SS and Theorem ICLT.

M83 (Robert Beezer) In light of Theorem MRCLT and Theorem IMR, write a short comparison of Theorem NPNT and Exercise IVLT.T40.

T20 (Robert Beezer) Construct a new solution to Exercise B.T50 along the following outline. From the $n \times n$ matrix A, construct the linear transformation $T\colon \mathbb{C}^n \to \mathbb{C}^n$, $T(\mathbf{x}) = A\mathbf{x}$. Use Theorem NI, Theorem IMILT and Theorem ILTIS to translate between the nonsingularity of A and the surjectivity/injectivity of T. Then apply Theorem ILTB and Theorem SLTB to connect these properties with bases.

Solution (Robert Beezer) Given the nonsingular $n \times n$ matrix A, create the linear transformation $T\colon \mathbb{C}^n \to \mathbb{C}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$. Then

$$\begin{aligned} A \text{ nonsingular} &\iff A \text{ invertible} && \text{Theorem NI} \\ &\iff T \text{ invertible} && \text{Theorem IMILT} \\ &\iff T \text{ injective and surjective} && \text{Theorem ILTIS} \\ &\iff C \text{ linearly independent, and} && \text{Theorem ILTB} \\ &\qquad C \text{ spans } \mathbb{C}^n && \text{Theorem SLTB} \\ &\iff C \text{ basis for } \mathbb{C}^n && \text{Definition B} \end{aligned}$$

T60 (Robert Beezer) Create an entirely different proof of Theorem IMILT that relies on Definition IVLT to establish the invertibility of T, and that relies on Definition MI to establish the invertibility of A.

T80 (Robert Beezer) Suppose that $T\colon U \to V$ and $S\colon V \to W$ are linear transformations, and that B, C and D are bases for U, V, and W. Using only Definition MR define matrix representations for T and S. Using these two definitions, and Definition MR, derive a matrix representation for the composition $S \circ T$ in terms of the entries of the matrices $M^T_{B,C}$ and $M^S_{C,D}$. Explain how you would use this result to *motivate a definition* for matrix multiplication that is strikingly similar to Theorem EMP.

Solution (Robert Beezer) Suppose that $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_m\}$, $C = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ and $D = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \ldots, \mathbf{w}_p\}$. For convenience, set $M = M^T_{B,C}$, $m_{ij} = [M]_{ij}$, $1 \le i \le n$, $1 \le j \le m$, and similarly, set $N = M^S_{C,D}$, $n_{ij} = [N]_{ij}$, $1 \le i \le p$, $1 \le j \le n$. We want to learn about the matrix representation of $S \circ T\colon V \to W$ relative to B and D. We will examine a single (generic) entry of this representation.

$$
\begin{aligned}
\left[M^{S\circ T}_{B,D}\right]_{ij} &= \left[\rho_D\left(\left(S \circ T\right)\left(\mathbf{u}_j\right)\right)\right]_i && \text{Definition MR}\\
&= \left[\rho_D\left(S\left(T\left(\mathbf{u}_j\right)\right)\right)\right]_i && \text{Definition LTC}\\
&= \left[\rho_D\left(S\left(\sum_{k=1}^{n} m_{kj}\mathbf{v}_k\right)\right)\right]_i && \text{Definition MR}\\
&= \left[\rho_D\left(\sum_{k=1}^{n} m_{kj}S\left(\mathbf{v}_k\right)\right)\right]_i && \text{Theorem LTLC}\\
&= \left[\rho_D\left(\sum_{k=1}^{n} m_{kj}\sum_{\ell=1}^{p} n_{\ell k}\mathbf{w}_\ell\right)\right]_i && \text{Definition MR}\\
&= \left[\rho_D\left(\sum_{k=1}^{n}\sum_{\ell=1}^{p} m_{kj}n_{\ell k}\mathbf{w}_\ell\right)\right]_i && \text{Property DVA}\\
&= \left[\rho_D\left(\sum_{\ell=1}^{p}\sum_{k=1}^{n} m_{kj}n_{\ell k}\mathbf{w}_\ell\right)\right]_i && \text{Property C}\\
&= \left[\rho_D\left(\sum_{\ell=1}^{p}\left(\sum_{k=1}^{n} m_{kj}n_{\ell k}\right)\mathbf{w}_\ell\right)\right]_i && \text{Property DSA}\\
&= \sum_{k=1}^{n} m_{kj}n_{ik} && \text{Definition VR}\\
&= \sum_{k=1}^{n} n_{ik}m_{kj} && \text{Property CMCN}\\
&= \sum_{k=1}^{n} \left[M^S_{C,D}\right]_{ik}\left[M^T_{B,C}\right]_{kj} && \text{Property CMCN}
\end{aligned}
$$

This formula for the entry of a matrix should remind you of Theorem EMP. However, while the theorem presumed we knew how to multiply matrices, the solution before us never uses any understanding of matrix products. It uses the definitions of vector and matrix representations, properties of linear transformations and vector spaces. So if we began a course by first discussing vector space, and then linear transformations between vector spaces, we could carry matrix representations into a *motivation* for a definition of matrix multiplication that is grounded in function composition. That is worth saying again — a definition of matrix representations of linear transformations *results* in a matrix product being the representation of a composition of linear transformations.

This exercise is meant to explain why many authors take the formula in Theorem EMP as their *definition* of matrix multiplication, and why it is a natural choice when the proper motivation is in place. If we first defined matrix multiplication in the style of Theorem EMP, then the above argument, followed by a simple application of the definition of matrix equality (Definition ME), would yield Theorem MRCLT.

Section CB
Change of Basis

C20 (Robert Beezer) In Example CBCV we computed the vector representation of $\mathbf{y}$ relative to C, $\rho_C(\mathbf{y})$, as an example of Theorem CB. Compute this same representation directly. In other words, apply Definition VR rather than Theorem CB.

C21 (Robert Beezer) Perform a check on Example MRCM by computing $M^Q_{B,D}$ directly. In other words, apply Definition MR rather than Theorem MRCB.

Solution (Robert Beezer) Apply Definition MR,

$$\begin{aligned}
&\rho_D\left(Q\left(\begin{bmatrix}5 & -3\\ -3 & -2\end{bmatrix}\right)\right) = \rho_D\left(19+14x-2x^2-28x^3\right)\\
&= \rho_D\left((-39)(2+x-2x^2+3x^3)+62(-1-2x^2+3x^3)+(-53)(-3-x+x^3)+(-44)(-x^2+x^3)\right)\\
&= \begin{bmatrix}-39\\ 62\\ -53\\ -44\end{bmatrix}\\
&\rho_D\left(Q\left(\begin{bmatrix}2 & -3\\ -3 & 0\end{bmatrix}\right)\right) = \rho_D\left(16+9x-7x^2-14x^3\right)\\
&= \rho_D\left((-23)(2+x-2x^2+3x^3)+(34)(-1-2x^2+3x^3)+(-32)(-3-x+x^3)+(-15)(-x^2+x^3)\right)\\
&= \begin{bmatrix}-23\\ 34\\ -32\\ -15\end{bmatrix}\\
&\rho_D\left(Q\left(\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\right)\right) = \rho_D\left(25+9x+3x^2+4x^3\right)\\
&= \rho_D\left((14)(2+x-2x^2+3x^3)+(-12)(-1-2x^2+3x^3)+5(-3-x+x^3)+(-7)(-x^2+x^3)\right)\\
&= \begin{bmatrix}14\\ -12\\ 5\\ -7\end{bmatrix}
\end{aligned}$$

These three vectors are the columns of the matrix representation,

$$M^Q_{B,D} = \begin{bmatrix}-39 & -23 & 14\\ 62 & 34 & -12\\ -53 & -32 & 5\\ -44 & -15 & -7\end{bmatrix}$$

which coincides with the result obtained in Example MRCM.

C30 (Robert Beezer) Find a basis for the vector space P_3 composed of eigenvectors of the linear transformation T. Then find a matrix representation of T relative to this basis.

$$T\colon P_3 \to P_3,\quad T\left(a+bx+cx^2+dx^3\right) = (a+c+d)+(b+c+d)x+(a+b+c)x^2+(a+b+d)x^3$$

Solution (Robert Beezer) With the domain and codomain being identical, we will build a matrix representation using the same basis for both the domain and codomain. The eigenvalues of the matrix representation will be the eigenvalues of the linear transformation, and we can obtain the eigenvectors of the linear transformation by un-coordinatizing (Theorem EER). Since the method does not depend on *which* basis we choose, we can choose a natural basis for ease of computation, say,

$$B = \left\{1,\, x,\, x^2, x^3\right\}$$

The matrix representation is then,

$$M^T_{B,B} = \begin{bmatrix}1 & 0 & 1 & 1\\ 0 & 1 & 1 & 1\\ 1 & 1 & 1 & 0\\ 1 & 1 & 0 & 1\end{bmatrix}$$

The eigenvalues and eigenvectors of this matrix were computed in Example ESMS4. A basis for $\mathbb{C}^4$, composed of eigenvectors of the matrix representation is,

$$C = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Applying ρ_B^{-1} to each vector of this set, yields a basis of P_3 composed of eigenvectors of T,

$$D = \left\{1 + x + x^2 + x^3, -1 + x, -x^2 + x^3, -1 - x + x^2 + x^3\right\}$$

The matrix representation of T relative to the basis D will be a diagonal matrix with the corresponding eigenvalues along the diagonal, so in this case we get

$$M^T_{D,D} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

C40 (Robert Beezer) Let S_{22} be the vector space of 2×2 symmetric matrices. Find a basis C for S_{22} that yields a diagonal matrix representation of the linear transformation R.

$$R\colon S_{22} \to S_{22}, \quad R\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = \begin{bmatrix} -5a + 2b - 3c & -12a + 5b - 6c \\ -12a + 5b - 6c & 6a - 2b + 4c \end{bmatrix}$$

Solution (Robert Beezer) Begin with a matrix representation of R, any matrix representation, but use the same basis for both instances of S_{22}. We'll choose a basis that makes it easy to compute vector representations in S_{22}.

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Then the resulting matrix representation of R (Definition MR) is

$$M^R_{B,B} = \begin{bmatrix} -5 & 2 & -3 \\ -12 & 5 & -6 \\ 6 & -2 & 4 \end{bmatrix}$$

Now, compute the eigenvalues and eigenvectors of this matrix, with the goal of diagonalizing the matrix (Theorem DC),

$$\lambda = 2 \qquad \mathcal{E}_{M^R_{B,B}}(2) = \left\langle \left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\} \right\rangle$$

$$\lambda = 1 \qquad \mathcal{E}_{M^R_{B,B}}(1) = \left\langle \left\{ \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \right\} \right\rangle$$

The three vectors that occur as basis elements for these eigenspaces will together form a linearly independent set (check this!). So these column vectors may be employed in a matrix that will diagonalize the matrix representation. If we "un-coordinatize" these three column vectors relative to the basis B, we will find three linearly independent elements of S_{22} that are eigenvectors of the linear transformation R (Theorem EER). A matrix representation relative to this basis of eigenvectors will be diagonal, with the eigenvalues ($\lambda = 2, 1$) as the diagonal elements. Here we go,

$$\rho_B^{-1}\left(\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right) = (-1)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-2)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}$$

$$\rho_B^{-1}\left(\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}\right) = (-1)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\rho_B^{-1}\left(\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}\right) = 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}$$

So the requested basis of S_{22}, yielding a diagonal matrix representation of R, is

$$C = \left\{ \begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix} \right\}$$

C41 (Robert Beezer) Let S_{22} be the vector space of 2×2 symmetric matrices. Find a basis for S_{22} composed of eigenvectors of the linear transformation $Q\colon S_{22} \to S_{22}$.

$$Q\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = \begin{bmatrix} 25a + 18b + 30c & -16a - 11b - 20c \\ -16a - 11b - 20c & -11a - 9b - 12c \end{bmatrix}$$

Solution (Robert Beezer) Use a single basis for both the domain and codomain, since they are equal.

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

The matrix representation of Q relative to B is

$$M = M^{Q}_{B,B} = \begin{bmatrix} 25 & 18 & 30 \\ -16 & -11 & -20 \\ -11 & -9 & -12 \end{bmatrix}$$

We can analyze this matrix with the techniques of Section EE and then apply Theorem EER. The eigenvalues of this matrix are $\lambda = -2,\ 1,\ 3$ with eigenspaces

$$\mathcal{E}_M(-2) = \left\langle \left\{ \begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix} \right\} \right\rangle \qquad \mathcal{E}_M(1) = \left\langle \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\} \right\rangle \qquad \mathcal{E}_M(3) = \left\langle \left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\} \right\rangle$$

Because the three eigenvalues are distinct, the three basis vectors from the three eigenspaces for a linearly independent set (Theorem EDELI). Theorem EER says we can uncoordinatize these eigenvectors to obtain eigenvectors of Q. By Theorem ILTLI the resulting set will remain linearly independent. Set

$$C = \left\{ \rho_B^{-1}\left(\begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix}\right), \rho_B^{-1}\left(\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right), \rho_B^{-1}\left(\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}\right) \right\} = \left\{ \begin{bmatrix} -6 & 4 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -3 & 2 \\ 2 & 1 \end{bmatrix} \right\}$$

Then C is a linearly independent set of size 3 in the vector space S_{22}, which has dimension 3 as well. By Theorem G, C is a basis of S_{22}.

T10 (Robert Beezer) Suppose that $T\colon V \to V$ is an invertible linear transformation with a nonzero eigenvalue λ. Prove that $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

Solution (Robert Beezer) Let $\mathbf{v}$ be an eigenvector of T for the eigenvalue λ. Then,

$$\begin{aligned} T^{-1}(\mathbf{v}) &= \frac{1}{\lambda}\lambda T^{-1}(\mathbf{v}) && \lambda \neq 0 \\ &= \frac{1}{\lambda}T^{-1}(\lambda\mathbf{v}) && \text{Theorem ILTLT} \\ &= \frac{1}{\lambda}T^{-1}(T(\mathbf{v})) && \mathbf{v}\text{ eigenvector of }T \\ &= \frac{1}{\lambda}I_V(\mathbf{v}) && \text{Definition IVLT} \\ &= \frac{1}{\lambda}\mathbf{v} && \text{Definition IDLT} \end{aligned}$$

which says that $\frac{1}{\lambda}$ is an eigenvalue of T^{-1} with eigenvector $\mathbf{v}$. Note that it is possible to prove that any eigenvalue of an invertible linear transformation is never zero. So the hypothesis that λ be nonzero is just a convenience for this problem.

T15 (Robert Beezer) Suppose that V is a vector space and $T\colon V \to V$ is a linear transformation. Prove that T is injective if and only if $\lambda = 0$ is not an eigenvalue of T.

Section OD
Orthonormal Diagonalization

T10 (Robert Beezer) Exercise MM.T35 asked you to show that AA^* is Hermitian. Prove directly that AA^* is a normal matrix.

T20 (Robert Beezer) In the discussion following Theorem OBNM we comment that the equation $\hat{\mathbf{x}} = U^*\mathbf{x}$ is just Theorem COB in disguise. Formulate this observation more formally and prove the equivalence.

Archetypes

This section contains definitions and capsule summaries for each archetypical example. Comprehensive and detailed analysis of each can be found in the online supplement.

Archetype A Linear system of three equations, three unknowns. Singular coefficient matrix with dimension 1 null space. Integer eigenvalues and a degenerate eigenspace for coefficient matrix.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 8 \\ x_1 + x_2 &= 5 \end{aligned}$$

Archetype B System with three equations, three unknowns. Nonsingular coefficient matrix. Distinct integer eigenvalues for coefficient matrix.

$$\begin{aligned} -7x_1 - 6x_2 - 12x_3 &= -33 \\ 5x_1 + 5x_2 + 7x_3 &= 24 \\ x_1 + 4x_3 &= 5 \end{aligned}$$

Archetype C System with three equations, four variables. Consistent. Null space of coefficient matrix has dimension 1.

$$\begin{aligned} 2x_1 - 3x_2 + x_3 - 6x_4 &= -7 \\ 4x_1 + x_2 + 2x_3 + 9x_4 &= -7 \\ 3x_1 + x_2 + x_3 + 8x_4 &= -8 \end{aligned}$$

Archetype D System with three equations, four variables. Consistent. Null space of coefficient matrix has dimension 2. Coefficient matrix identical to that of Archetype E, vector of constants is different.

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 8 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= -12 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 4 \end{aligned}$$

Archetype E System with three equations, four variables. Inconsistent. Null space of coefficient matrix has dimension 2. Coefficient matrix identical to that of Archetype D, constant vector is different.

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 2 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= 3 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 2 \end{aligned}$$

Archetype F System with four equations, four variables. Nonsingular coefficient matrix. Integer eigenvalues, one has "high" multiplicity.

$$\begin{aligned} 33x_1 - 16x_2 + 10x_3 - 2x_4 &= -27 \\ 99x_1 - 47x_2 + 27x_3 - 7x_4 &= -77 \\ 78x_1 - 36x_2 + 17x_3 - 6x_4 &= -52 \\ -9x_1 + 2x_2 + 3x_3 + 4x_4 &= 5 \end{aligned}$$

Archetype G System with five equations, two variables. Consistent. Null space of coefficient matrix has dimension 0. Coefficient matrix identical to that of Archetype H, constant vector is different.

$$\begin{aligned} 2x_1 + 3x_2 &= 6 \\ -x_1 + 4x_2 &= -14 \\ 3x_1 + 10x_2 &= -2 \\ 3x_1 - x_2 &= 20 \\ 6x_1 + 9x_2 &= 18 \end{aligned}$$

Archetype H System with five equations, two variables. Inconsistent, overdetermined. Null space of coefficient matrix has dimension 0. Coefficient matrix identical to that of Archetype G, constant vector is different.

$$\begin{aligned} 2x_1 + 3x_2 &= 5 \\ -x_1 + 4x_2 &= 6 \\ 3x_1 + 10x_2 &= 2 \\ 3x_1 - x_2 &= -1 \\ 6x_1 + 9x_2 &= 3 \end{aligned}$$

Archetype I System with four equations, seven variables. Consistent. Null space of coefficient matrix has dimension 4.

$$\begin{aligned} x_1 + 4x_2 - x_4 + 7x_6 - 9x_7 &= 3 \\ 2x_1 + 8x_2 - x_3 + 3x_4 + 9x_5 - 13x_6 + 7x_7 &= 9 \\ 2x_3 - 3x_4 - 4x_5 + 12x_6 - 8x_7 &= 1 \\ -x_1 - 4x_2 + 2x_3 + 4x_4 + 8x_5 - 31x_6 + 37x_7 &= 4 \end{aligned}$$

Archetype J System with six equations, nine variables. Consistent. Null space of coefficient matrix has dimension 5.

$$\begin{aligned} x_1 + 2x_2 - 2x_3 + 9x_4 + 3x_5 - 5x_6 - 2x_7 + x_8 + 27x_9 &= -5 \\ 2x_1 + 4x_2 + 3x_3 + 4x_4 - x_5 + 4x_6 + 10x_7 + 2x_8 - 23x_9 &= 18 \\ x_1 + 2x_2 + x_3 + 3x_4 + x_5 + x_6 + 5x_7 + 2x_8 - 7x_9 &= 6 \\ 2x_1 + 4x_2 + 3x_3 + 4x_4 - 7x_5 + 2x_6 + 4x_7 - 11x_9 &= 20 \\ x_1 + 2x_2 + 5x_4 + 2x_5 - 4x_6 + 3x_7 + 8x_8 + 13x_9 &= -4 \\ -3x_1 - 6x_2 - x_3 - 13x_4 + 2x_5 - 5x_6 - 4x_7 + 13x_8 + 10x_9 &= -29 \end{aligned}$$

Archetype K Square matrix of size 5. Nonsingular. 3 distinct eigenvalues, 2 of multiplicity 2.

$$\begin{bmatrix} 10 & 18 & 24 & 24 & -12 \\ 12 & -2 & -6 & 0 & -18 \\ -30 & -21 & -23 & -30 & 39 \\ 27 & 30 & 36 & 37 & -30 \\ 18 & 24 & 30 & 30 & -20 \end{bmatrix}$$

Archetype L Square matrix of size 5. Singular, nullity 2. 2 distinct eigenvalues, each of “high” multiplicity.

$$\begin{bmatrix} -2 & -1 & -2 & -4 & 4 \\ -6 & -5 & -4 & -4 & 6 \\ 10 & 7 & 7 & 10 & -13 \\ -7 & -5 & -6 & -9 & 10 \\ -4 & -3 & -4 & -6 & 6 \end{bmatrix}$$

Archetype M Linear transformation with bigger domain than codomain, so it is guaranteed to not be injective. Happens to not be surjective.

$$T\colon \mathbb{C}^5 \to \mathbb{C}^3, \quad T\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}\right) = \begin{bmatrix}x_1+2x_2+3x_3+4x_4+4x_5\\3x_1+x_2+4x_3-3x_4+7x_5\\x_1-x_2-5x_4+x_5\end{bmatrix}$$

Archetype N Linear transformation with domain larger than its codomain, so it is guaranteed to not be injective. Happens to be onto.

$$T\colon \mathbb{C}^5 \to \mathbb{C}^3, \quad T\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}\right) = \begin{bmatrix}2x_1+x_2+3x_3-4x_4+5x_5\\x_1-2x_2+3x_3-9x_4+3x_5\\3x_1+4x_3-6x_4+5x_5\end{bmatrix}$$

Archetype O Linear transformation with a domain smaller than the codomain, so it is guaranteed to not be onto. Happens to not be one-to-one.

$$T\colon \mathbb{C}^3 \to \mathbb{C}^5, \quad T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}-x_1+x_2-3x_3\\-x_1+2x_2-4x_3\\x_1+x_2+x_3\\2x_1+3x_2+x_3\\x_1+2x_3\end{bmatrix}$$

Archetype P Linear transformation with a domain smaller that its codomain, so it is guaranteed to not be surjective. Happens to be injective.

$$T\colon \mathbb{C}^3 \to \mathbb{C}^5, \quad T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}-x_1+x_2+x_3\\-x_1+2x_2+2x_3\\x_1+x_2+3x_3\\2x_1+3x_2+x_3\\-2x_1+x_2+3x_3\end{bmatrix}$$

Archetype Q Linear transformation with equal-sized domain and codomain, so it has the potential to be invertible, but in this case is not. Neither injective nor surjective. Diagonalizable, though.

$$T\colon \mathbb{C}^5 \to \mathbb{C}^5, \quad T\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}\right) = \begin{bmatrix}-2x_1+3x_2+3x_3-6x_4+3x_5\\-16x_1+9x_2+12x_3-28x_4+28x_5\\-19x_1+7x_2+14x_3-32x_4+37x_5\\-21x_1+9x_2+15x_3-35x_4+39x_5\\-9x_1+5x_2+7x_3-16x_4+16x_5\end{bmatrix}$$

Archetype R Linear transformation with equal-sized domain and codomain. Injective, surjective, invertible, diagonalizable, the works.

$$T\colon \mathbb{C}^5 \to \mathbb{C}^5, \quad T\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}\right) = \begin{bmatrix}-65x_1+128x_2+10x_3-262x_4+40x_5\\36x_1-73x_2-x_3+151x_4-16x_5\\-44x_1+88x_2+5x_3-180x_4+24x_5\\34x_1-68x_2-3x_3+140x_4-18x_5\\12x_1-24x_2-x_3+49x_4-5x_5\end{bmatrix}$$

Archetype S Domain is column vectors, codomain is matrices. Domain is dimension 3 and codomain is dimension 4. Not injective, not surjective.

$$T\colon \mathbb{C}^3 \to M_{22}, \quad T\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = \begin{bmatrix}a-b & 2a+2b+c\\3a+b+c & -2a-6b-2c\end{bmatrix}$$

Archetype T Domain and codomain are polynomials. Domain has dimension 5, while codomain has dimension 6. Is injective, can't be surjective.

$$T\colon P_4 \to P_5, \quad T\left(p(x)\right) = (x-2)p(x)$$

Archetype U Domain is matrices, codomain is column vectors. Domain has dimension 6, while codomain has dimension 4. Can't be injective, is surjective.

$$T\colon M_{23}\to\mathbb{C}^4,\quad T\left(\begin{bmatrix}a & b & c\\ d & e & f\end{bmatrix}\right)=\begin{bmatrix}a+2b+12c-3d+e+6f\\ 2a-b-c+d-11f\\ a+b+7c+2d+e-3f\\ a+2b+12c+5e-5f\end{bmatrix}$$

Archetype V Domain is polynomials, codomain is matrices. Both domain and codomain have dimension 4. Injective, surjective, invertible. Square matrix representation, but domain and codomain are unequal, so no eigenvalue information.

$$T\colon P_3\to M_{22},\quad T\left(a+bx+cx^2+dx^3\right)=\begin{bmatrix}a+b & a-2c\\ d & b-d\end{bmatrix}$$

Archetype W Domain is polynomials, codomain is polynomials. Domain and codomain both have dimension 3. Injective, surjective, invertible, 3 distinct eigenvalues, diagonalizable.

$$\begin{aligned}&T\colon P_2\to P_2,\\ &T\left(a+bx+cx^2\right)=(19a+6b-4c)+(-24a-7b+4c)\,x+(36a+12b-9c)\,x^2\end{aligned}$$

Archetype X Domain and codomain are square matrices. Domain and codomain both have dimension 4. Not injective, not surjective, not invertible, 3 distinct eigenvalues, diagonalizable.

$$T\colon M_{22}\to M_{22},\quad T\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right)=\begin{bmatrix}-2a+15b+3c+27d & 10b+6c+18d\\ a-5b-9d & -a-4b-5c-8d\end{bmatrix}$$